SOME FUN WITH PRIMES
USING FOURIER SERIES TO SIEVE INTEGERS

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Abstract. This paper develops an exact count function for the number of primes that are the upper prime of a pair of twinned primes.

The method used is a modified Sieve of Eratosthenes. The strand functions of the modified sieve remove integers which are congruent to either 0 or 2 for some modulus, P_i. The strand functions for each of the strands of the sieve are represented by a Fourier series which converges almost everywhere to the strand function. The strand functions for the r strands of the sieve are combined to form a member function for the whole sieve. The member function of the sieve is then integrated to create the count function of the number of sieve survivors less than or equal to x. The sieve count function can then be evaluated at a paticular value of x, and all sieve survivors in the range: $1 < x < P_{r+1}^2 - 2$, are the upper prime of a pair of twinned primes.

With the count function defined, the lower bound on the value of the count function is derived. Using this lower bound it is possible to prove that for all $r \geq 12$ there exists at least one integer in the range, $1 < x < {P_{r+1}}^2 - 2$ which survives the sieve constructed. Such a sieve survivor is necessarily the greater prime of a pair of twinned primes and, because r may increase without bound, there must be an infinite number of primes which are the greater prime of a pair of twinned primes.

Date: Revised on: July 21, 2010.

2000 *Mathematics Subject Classification.* Primary 11A41; Secondary 11L03,11L07.

Key words and phrases. Analytic Number Theory,Fourier Series, Distribution of Primes,Twinned Primes,Twin Primes.

Thanks to my patient wife, Lori, who has endured nearly 3 decades of this madness.

1. Introduction

The sieve of Eratosthenes is an elementary method for finding primes. Generalizing slightly one can introduce the following ideas:

- Strands to the sieve,
- The basis for each strand in the sieve,
- The modulus of the sieve, and
- Survivors of the sieve.

A strand can be best defined as a linear equation over a modulus. The modulus is the basis of strand. A sieve is a system strands where the basis of each strand is relatively prime to the basis of every other strand in the sieve. The modulus of the sieve is the product of the several basis of each strand of the sieve. A survivor of the sieve is an integer that is not sieved by any of the strands of the sieve. In this paper the basis for every strand is a prime number. Specifically, if the sieve has r strands the basis of the strands are the first r primes; $2, 3, 5, \ldots, P_r$

A simple 5-strand sieve will help to illustrate these concepts. Since this paper is ultimately about twinned primes, the example sieve will not sieve for prime numbers, but instead will sieve for the the upper prime of a pair of twinned primes. Two numbers, x and $x+2$, are prime if both x and $x+2$ are relatively prime to every prime less than or equal to equal to $\sqrt{x}$. This can be represented by a system of Diophantine equations.

In order for an integer n to be the upper prime of a pair of twinned primes

- $n \leq P_{r+1}^2 - 2$
- $n \neq 0 \pmod{P_i}$ for all $1 \leq i \leq r$, and
- $n - 2 \neq 0 \pmod{P_i}$ for all $1 \leq i \leq r$.

2. Formulating the Count Function

Another way to represent these conditions is with a system of Diophantine equations:

$$n = 1 \pmod 2 \tag{1a}$$
$$n \neq \{0, 2\} \pmod 3 \tag{1b}$$
$$n \neq \{0, 2\} \pmod 5 \tag{1c}$$
$$n \neq \{0, 2\} \pmod 7 \tag{1d}$$
$$\vdots \tag{1e}$$
$$n \neq \{0, 2\} \pmod{P_{r-2}} \tag{1f}$$
$$n \neq \{0, 2\} \pmod{P_{r-1}} \tag{1g}$$
$$n \neq \{0, 2\} \pmod{P_r} \tag{1h}$$

Each of the discrete,integer equations in (1) can be represented as discontinuous periodic functions:

$$f_1(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases} \tag{2a}$$

$$f_2(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 3 \end{cases} \tag{2b}$$

$$f_3(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \\ 1 & 2 < x < 4 \\ 0 & 4 < x < 5 \end{cases} \tag{2c}$$

$$f_4(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \\ 1 & 2 < x < 6 \\ 0 & 6 < x < 7 \end{cases} \tag{2d}$$

$$f_5(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \\ 1 & 2 < x < 10 \\ 0 & 10 < x < 11 \end{cases} \tag{2e}$$

$$\vdots \tag{2f}$$

$$f_{r-2}(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \\ 1 & 2 < x < P_{r-2} - 1 \\ 0 & P_{r-2} - 1 < x < P_{r-2} \end{cases} \tag{2g}$$

$$f_{r-1}(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \\ 1 & 2 < x < P_{r-1} - 1 \\ 0 & P_{r-1} - 1 < x < P_{r-1} \end{cases} \tag{2h}$$

$$f_r(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \\ 1 & 2 < x < P_r - 1 \\ 0 & P_r - 1 < x < P_r \end{cases} \tag{2i}$$

where the strand functions are extended over the whole real line by the periodic relations:

$$f_1(x+2n) = f_1(x) \qquad \forall n \in \mathbb{Z} \quad 0 \leq x < 2 \tag{3a}$$
$$f_2(x+3n) = f_2(x) \qquad \forall n \in \mathbb{Z} \quad 0 \leq x < 3 \tag{3b}$$
$$f_3(x+5n) = f_3(x) \qquad \forall n \in \mathbb{Z} \quad 0 \leq x < 5 \tag{3c}$$
$$f_4(x+7n) = f_4(x) \qquad \forall n \in \mathbb{Z} \quad 0 \leq x < 7 \tag{3d}$$
$$f_5(x+11n) = f_5(x) \qquad \forall n \in \mathbb{Z} \quad 0 \leq x < 11 \tag{3e}$$
$$\vdots \qquad \vdots \tag{3f}$$
$$f_{r-2}(x+P_{r-2}n) = f_{r-2}(x) \qquad \forall n \in \mathbb{Z} \quad 0 \leq x < P_{r-2} \tag{3g}$$
$$f_{r-1}(x+P_{r-1}n) = f_{r-1}(x) \qquad \forall n \in \mathbb{Z} \quad 0 \leq x < P_{r-1} \tag{3h}$$
$$f_r(x+P_rn) = f_r(x) \qquad \forall n \in \mathbb{Z} \quad 0 \leq x < P_r \tag{3i}$$

The strand functions are periodic modulo P_j , which is the basis of the strand. The strand functions have a value of either 1 or 0. The strand functions are undefined at the discontinuities which mark the transitions between 0 and 1. The strand functions were defined in this manner so that the integral of an individual strand function would yield a proper counting function for the individual strand of the sieve. If one of the strand functions is integrated as follows:

$$F_j(x) = \int_{t=0}^{x} f_j(t)dt \quad \text{where: } 1 \leq j \leq r$$

the result is

$$F_1(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$$\text{exended by:} \quad F_1(x) = F_1\left(x - 2\left\lfloor\frac{x}{2}\right\rfloor\right) + \left\lfloor\frac{x}{2}\right\rfloor$$

$$F_2(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 3 \end{cases}$$

$$\text{exended by:} \quad F_2(x) = F_2\left(x - 3\left\lfloor\frac{x}{3}\right\rfloor\right) + \left\lfloor\frac{x}{3}\right\rfloor$$

$$F_3(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \\ x-1 & 2 < x < 4 \\ 0 & 4 < x < 5 \end{cases}$$

$$\text{exended by:} \quad F_3(x) = F_3\left(x - 5\left\lfloor\frac{x}{5}\right\rfloor\right) + 3\left\lfloor\frac{x}{5}\right\rfloor$$

$$F_4(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \\ x-1 & 2 < x < 6 \\ 0 & 6 < x < 7 \end{cases}$$

$$\text{exended by:} \quad F_4(x) = F_4\left(x - 7\left\lfloor\frac{x}{7}\right\rfloor\right) + 5\left\lfloor\frac{x}{7}\right\rfloor$$

$$F_{r-1}(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \\ x-1 & 2 < x < P_{r-1} \\ P_{r-1}-2 & P_{r-1}-1 < x < P_{r-1} \end{cases}$$

$$\text{exended by:} \quad F_{r-1}(x) = F_{r-1}\left(x - P_{r-1}\left\lfloor\frac{x}{P_{r-1}}\right\rfloor\right) + (P_{r-1}-2)\left\lfloor\frac{x}{P_{r-1}}\right\rfloor$$

$$F_r(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \\ x-1 & 2 < x < P_r \\ P_r-2 & P_r-1 < x < P_r \end{cases}$$

$$\text{exended by:} \quad F_r(x) = F_r\left(x - P_r\left\lfloor\frac{x}{P_r}\right\rfloor\right) + (P_r-2)\left\lfloor\frac{x}{P_r}\right\rfloor$$

The count function for a particular strand ramps up from N to $N+1$ in the intervals where the strand function is equal to 1. The count function for a particular strand remains constant for the intervals where the strand function is equal to 0. The function, $F_r(x)$, is the count function for the strand $f_r(t)$. This is because for all integer values, n, $F_r(n)$ is the number of integers less than or equal to n which survive the strand function, $f_r(t)$. For all real values of x, $\lfloor F_r(x) \rfloor$ is the number of integers less than or equal to x that survive the strand function, $f_r(t)$.

Using the strand functions defined above it is possible to create a survivor function, $\pi_2(r,t)$, for a sieve of r strands. The strand functions are multiplied together and, neglecting for the moment the values at the points of discontinuity, the product of the first r strand functions is the survivor function of the r-strand sieve. The sieve survivor function is 0 everywhere except for unit intervals where the function is equal to 1. The unit intervals where the survivor function is 1 are exactly those intervals of $n-1 < x < n$ where n is an integer that survives all the strand functions of the sieve.

The product of the first, r strand functions is:

$$\pi_2(r,t) = \prod_{j=1}^{r} f_j(t)$$

The function, $\pi_2(r,t)$, has a value of either 0 or 1. The function, $\pi_2(r,t)$, is 1 in the interval $n-1 < x < n$, if and only if n is an integer which survives the first r strand functions. The function, $\pi_2(r,t)$, is 0 in all the intervals $n-1 < x < n$. The function, $\pi_2(r,t)$, is undefined for $x \in \mathbb{Z}$; the transitions between 0 and 1. Integrating the membership function, $\pi_2(r,t)$, of the sieve yields the counting function:

$$\begin{aligned} T\left(r,x\right) &= \int_{t=0}^{x} \pi_2(r,t)dt \\ &= \int_{t=0}^{x} \left[\prod_{j=1}^{r} f_j(t)\right] dt \end{aligned}$$

Integrating a product of several functions is difficult. This definition of the sieve does not seem to be much of an improvement. Fortunately, each of the strand functions meets Dirichlet's conditions. Because of this, for every strand functions there exists a Fourier representation which converges almost everywhere to the strand function. At the points of discontinuity the Fourier series representation of the strand function converges to 1/2. Thus, every strand function can be represented by a Fourier series of the form

$$f_j(t) = \sum_{n_j \in \mathbb{Z}} C(j, n_j)\, e^{2\pi i \frac{n_j}{P_j} t}$$

where:

$$C(j, n_j) = \frac{1}{P_j} \int_{t=0}^{P_j} f_j(t) e^{-2\pi i \frac{n_j}{P_j} t} dt$$

Evaluating the above integrals for $j = 1$, for $j = 2$ and for $j > 2$ yields:

$$C(1, n_1) = \frac{1}{2} \int_{t=0}^{1} e^{-2\pi i \frac{n_1}{2} t} dt \tag{4a}$$

$$C(2, n_2) = \frac{1}{3} \int_{t=0}^{1} e^{-2\pi i \frac{n_2}{3} t} dt \tag{4b}$$

$$C(j, n_j) = \frac{1}{P_j} \int_{t=0}^{1} e^{-2\pi i \frac{n_j}{P_j} t} dt \tag{4c}$$

$$+ \frac{1}{P_j} \int_{t=2}^{P_j - 1} e^{-2\pi i \frac{n_j}{P_j} t} dt \tag{4d}$$

Separating out the case where $n_j = 0$ yields:

$$C(1, 0) = \frac{1}{2} \tag{5a}$$

$$C(2, 0) = \frac{1}{3} \tag{5b}$$

$$C(j, 0) = 1 - \frac{2}{P_j} \tag{5c}$$

Separating out the case where $n_j \neq 0$ yields:

$$C\left(1,n_1\right) = \frac{1-e^{-2\pi i \frac{n_1}{2}}}{2\pi i n_1} = \frac{1-\left(-1\right)^{n_1}}{2\pi i n_1} \tag{6a}$$

$$C\left(2,n_2\right) = \frac{1-e^{-2\pi i \frac{n_2}{3}}}{2\pi i n_2} \tag{6b}$$

$$C\left(j,n_j\right) = \frac{\left(1+e^{-4\pi i \frac{n_j}{P_j}}\right)\left(1-e^{2\pi i \frac{n_j}{P_j}}\right)}{2\pi i n_j} \tag{6c}$$

Some properties of the Fourier constants, $C\left(j,n_j\right)$, should be noted immediately.

(1) $C\left(j,-n_j\right) = \overline{C}\left(j,n_j\right)$ where $\overline{C}\left(j,n_j\right)$ is the complex conjugate of $C\left(j,n_j\right)$.
(2) $C\left(j,P_j n_j\right) = 0$ for all indices, $n_j \neq 0$, and $1 \leq j \leq r$.
(3) $C\left(j,k_j+nP_j\right) = C\left(j,k_j\right)\frac{k_j}{\left(k_j+nP_j\right)}$ for all $n \in \mathbb{Z}$.

Substituting the Fourier series representation for $f_j\left(t\right)$ into the definition of $T\left(r,x\right)$ yields:

$$T\left(r,x\right)=\int\limits_{t=0}^{x}\left[\prod_{j=1}^{r}\left[\sum_{n_j\in\mathbb{Z}}C\left(j,n_j\right)e^{2\pi i\frac{n_j}{P_j}t}\right]\right]dt \tag{7a}$$

$$T\left(r,x\right)=\int\limits_{t=0}^{x}\left[\sum_{n_1\in\mathbb{Z}}C\left(1,n_1\right)e^{2\pi i\frac{n_1}{2}t}\right]\left[\sum_{n_2\in\mathbb{Z}}C\left(2,n_2\right)e^{2\pi i\frac{n_2}{3}t}\right]\left[\sum_{n_3\in\mathbb{Z}}C\left(3,n_3\right)e^{2\pi i\frac{n_3}{5}t}\right]\dots\left[\sum_{n_r\in\mathbb{Z}}C\left(r,n_r\right)e^{2\pi i\frac{n_j}{P_r}t}\right]dt \tag{7b}$$

$$T\left(r,x\right)=\int\limits_{t=0}^{x}\sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}}\dots\sum_{n_r\in\mathbb{Z}}\left[C\left(1,n_1\right)e^{2\pi i\frac{n_1}{2}t}\right]\left[C\left(2,n_2\right)e^{2\pi i\frac{n_2}{3}t}\right]\left[C\left(3,n_3\right)e^{2\pi i\frac{n_3}{5}t}\right]\dots\left[C\left(r,n_r\right)e^{2\pi i\frac{n_r}{P_r}t}\right]dt \tag{7c}$$

$$T\left(r,x\right)=\int\limits_{t=0}^{x}\sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}}\dots\sum_{n_r\in\mathbb{Z}}\left[\prod_{j=1}^{r}C\left(j,n_j\right)\right]\left[e^{2\pi it\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right] \tag{7d}$$

$$T\left(r,x\right)=\sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}}\dots\sum_{n_r\in\mathbb{Z}}\left[\prod_{j=1}^{r}C\left(j,n_j\right)\right]\left[\int\limits_{t=0}^{x}e^{2\pi it\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}dt\right] \tag{7e}$$

Equation (7e) can be separated into 3 cases:

Case 1: $\sum\limits_{j=1}^{r} \frac{n_j}{P_j} = 0$ because all $n_j = 0$

Case 2: $\sum\limits_{j=1}^{r} \frac{n_j}{P_j} = 0$ and at least $n_j \neq 0$

Case 3: $\sum\limits_{j=1}^{r} \frac{n_j}{P_j} \neq 0$ because at least one $n_j \neq 0$

For case 2, $\sum\limits_{j=1}^{r} \frac{n_j}{P_j} = 0$, if and only if, for all indices (n_j; where $1 \leq j \leq r$) either $P_j | n_j$ or $n_j = 0$. If $P_j | n_j$, then $C(j, n_j) = 0$ and $\prod\limits_{j=1}^{r} C(j, n_j) = 0$. Separating out the remaining two cases yields:

$$T(r,x) = \int\limits_{t=0}^{x} \prod_{j=1}^{r} C(j,0)\, dt + \sum_{\substack{n_1 \in \mathbb{Z}}} \sum_{\substack{n_2 \in \mathbb{Z} \\ \sum\limits_{j=1}^{r} \frac{n_j}{P_j} \neq 0}} \sum_{n_3 \in \mathbb{Z}} \cdots \sum_{n_r \in \mathbb{Z}} \left[\prod_{j=1}^{r} C(j, n_j) \right] \left[\int\limits_{t=0}^{x} e^{2\pi i t \sum\limits_{j=1}^{r} \frac{n_j}{P_j}} dt \right] \tag{8}$$

$$T(r,x) = x \prod_{j=1}^{r} C(j,0) + \sum_{\substack{n_1 \in \mathbb{Z}}} \sum_{\substack{n_2 \in \mathbb{Z} \\ \sum\limits_{j=1}^{r} \frac{n_j}{P_j} \neq 0}} \sum_{n_3 \in \mathbb{Z}} \cdots \sum_{n_r \in \mathbb{Z}} \left[\prod_{j=1}^{r} C(j, n_j) \right] \left[\frac{e^{2\pi i x \sum\limits_{j=1}^{r} \frac{n_j}{P_j}} - 1}{2\pi i \sum\limits_{j=1}^{r} \frac{n_j}{P_j}} \right] \tag{9}$$

The expression above for $T(r,x)$, is not be pretty, but it is computable. For any value of x, the integer portion of the value of $T(r,x)$ evaluated at x and r is equal to the number of integers which are less than or equal to x which survive the r-strand sieve. If the survivor is in the range, $1 < x < P_{r+1}^2 - 2$, then, by the nature of the sieve, the survivor is the upper prime of a pair twinned primes. The challenge, as with all sieve methods, is to understand the behavior and bounds of $T(r,x)$.

The first qualitative observation is that the count function has two components; the average growth and the variation about this average growth. The average growth term is given by:

$$A(r,x) = x\prod_{j=1}^{r} C\left(j,0\right) \tag{10}$$

and the variation component is given by:

$$V(r,x) = \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{\substack{n_3\in\mathbb{Z} \\ \sum_{j=1}^{r}\frac{n_j}{P_j}\neq 0}}\cdots\sum_{n_r\in\mathbb{Z}}\left[\prod_{j=1}^{r} C\left(j,n_j\right)\right]\left[\frac{e^{2\pi i x\sum_{j=1}^{r}\frac{n_j}{P_j}}-1}{2\pi i\sum_{j=1}^{r}\frac{n_j}{P_j}}\right] \tag{11}$$

Other qualitative properties of $T\left(r,x\right)$, $A\left(r,x\right)$, and $V\left(r,x\right)$, are:

(1) $T\left(r,x\right)$ is a monotone, non-decreasing function. If $y \geq x$, then $T\left(r,y\right) \geq T\left(r,x\right)$.
(2) If m is a integer which survives the r-stranded sieve and n is the number of integers $\leq m$ which survive the sieve, the value of $T\left(r,x\right)$ ramps up from n to $n+1$ on the interval $m \leq x \leq m+1$. i.e. $T\left(r,m\right) = n$ and $T\left(r,m+1\right) = n+1$
(3) $\lfloor T\left(r,x\right)\rfloor$ is equal to the number of integers less than or equal to x which survive the sieve.
(4) $V(r,x)$ is periodic.
(5) The period of $V(r,x)$ is equal to the modulus of the sieve, M_r; where $M_r = \prod_{j=1}^{r} P_j$.
(6) The minima and maxima of $V(r,x)$ occur when $x \in \mathbb{Z}$.
(7) $V(r,x)$ is a series of connect line segments such that
$$V(r,x) = V(r,\lfloor x\rfloor) + (x-\lfloor x\rfloor)\left[V(r,\lceil x\rceil) - V(r,\lfloor x\rfloor)\right]$$
(8) For all $r \geq 1$, $\sum_{j=1}^{r}\frac{n_j}{P_j} \in \{\mathbb{Q}/\mathbb{Z}\}\cup\{0\}$. In other words, except for $\sum_{j=1}^{r}\frac{n_j}{P_j} = 0$ because all $n_j = 0$, $\sum_{j=1}^{r}\frac{n_j}{P_j} \notin \mathbb{Z}$. Moreover, $\left(\sum_{j=1}^{K}\frac{n_j}{P_j}\right) \pm \left(\sum_{j=K+1}^{r}\frac{n_j}{P_j}\right) \notin \mathbb{Z}$

3. Notational Conventions

In this paper all variables, except P, x, and t, are assumed to be arbitrary integers. The variables P_j are prime numbers where $P_1 = 2$, $P_2 = 3$, $P_3 = 5$, $P_4 = 7$, etc. The variables x and t are assumed to be real. The function, $T(r,x)$, is used in this paper to designate the number of integers less than or equal to x which survive the r-strand sieve. The more conventional notation for this count function would be $\pi_2(r,x)$. This presents much confusion as the constant, π, is used in both the complex exponentials and in the Fourier constants. It is hoped the less conventional notation, $T(r,x)$, will be less confusing.

The notation

$$\underbrace{\sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\sum_{n_3\in\mathbb{Z}}\cdots\sum_{n_{r-1}\in\mathbb{Z}}\sum_{n_r\in\mathbb{Z}}}_{\sum_{j=1}^{r}\frac{n_j}{P_j}\neq 0} f(n_1,n_2,n_3,\ldots,n_{r-1},n_r)$$

is cumbersome and will be replaced by:

$$\dot{\sum_{\vec{n_r}\in\mathbb{Z}}} f(n_1,n_2,n_3,\ldots,n_{r-1},n_r)$$

The vector, $\vec{n_r}$, is an r-tuple of integers and it is understood any number of, but not all, indices n_j, can equal zero. The r-tuple, $\vec{n_r}$, formed by the r indices, $(n_1,n_2,n_3,\ldots,n_{r-1},n_r)$, cannot equal the r-tuple: $(0,0,0,\ldots,0,0)$.

The count function (9) can then be written more compactly as:

$$T(r,x) = x\prod_{j=1}^{r} C(j,0) + \dot{\sum_{\vec{n_r}\in\mathbb{Z}}}\left[\prod_{j=1}^{r} C(j,n_j)\right]\left[\frac{e^{2\pi i x\sum_{j=1}^{r}\frac{n_j}{P_j}}-1}{2\pi i\sum_{j=1}^{r}\frac{n_j}{P_j}}\right] \tag{12}$$

For a summation over a single index the notation,

$$\dot{\sum_{n_j\in\mathbb{Z}}} f(n_j)$$

is equivalent to

$$\sum_{\substack{n_j\in\mathbb{Z}\\ n_j\neq 0}} f(n_j)$$

.

4. Main Proof

There is an infinite number of primes such that both P and $P-2$ are prime numbers.

Proof. From the definition of $T(r,x)$ in equation (12)

$$T(r,x) = x\prod_{j=1}^{r} C(j,0) + \dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}} \left[\prod_{j=1}^{r} C(j,n_j)\right] \left[\frac{e^{2\pi i x\sum\limits_{j=1}^{r}\frac{n_j}{P_j}} - 1}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right] \tag{13}$$

for all $r>0$ and all $x>0$.

Thus

$$T(r,x) \geq x\prod_{j=1}^{r} C(j,0) - \left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}} \left[\prod_{j=1}^{r} C(j,n_j)\right] \frac{e^{2\pi i x\sum\limits_{j=1}^{r}\frac{n_j}{P_j}} - 1}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right| \tag{14}$$

From Lemma 3, for real $x\geq 2$ and all $r>5$ $(P_r>11)$,

$$\left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}} \left[\prod_{j=1}^{r} C(j,n_j)\right] \frac{e^{2\pi i x\sum\limits_{j=1}^{r}\frac{n_j}{P_j}} - 1}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right| < 2r \tag{15}$$

This yields:

$$T(r,x) > x\prod_{j=1}^{r} C(j,0) - 2r \tag{16}$$

Evaluating above inequality, (15), at P_r^2 yields,

$$T\left(r,P_r^2\right) > P_r^2\prod_{j=1}^{r} C(j,0) - 2r \tag{17}$$

From Lemma 1,

$$P_r^2\prod_{j=1}^{r} C(j,0) > P_r + 2 \tag{18}$$

for all $r\geq 12$

Connecting,

- Inequality (18),
- Inequality (17), and
- $P_r > 2r$ for all $r\geq 3$

yields:

$$\begin{aligned} T\left(r, P_r^2\right) &> (P_r + 2) - 2r \\ &> (P_r - 2r) + 2 \\ &> 2 \end{aligned} \tag{19}$$

for all $x > 2$ and all $r \geq 12$ ($P_{12} = 37$).

For all $x > 2$ and all $r \geq 12$, $T\left(r, P_r^2\right) > 2$; i.e. there are two integers which survive the r-stranded sieve. One of the 2 survivors of the sieve is the integer 1. The remaining integer which survives the r-strand sieve, is less than P_r^2. By the design of the sieve, all of the integers which survive the r-stranded sieve, and are in the range: $1 < x < P_{r+1}^2 - 2$, must necessarily be the upper prime of a pair of twinned primes. Thus, the other sieve survivor of $T\left(r, P_r^2\right)$ is the upper prime of a pair of twinned primes.

Since r can increase without bound there must be an infinite number prime numbers which are the upper prime of a pair of twinned primes.

□

5. Lemma 1

Lemma 1.

$$P_r^2 \prod_{j=1}^{r} C\left(j,0\right) - P_r > 2 \tag{20}$$

for all $x \geq 0$ *and all* $r \geq 12$

Proof. The proof by induction has to steps:

- Prove that the propostion being true for r implies the proposition is true for $r+2$
- Prove the propostion is actually true for two successive values of r.

From Lemma 2

$$\left(\frac{P_{r+2}-2}{P_{r+1}}\right)\left(\frac{P_{r+1}-2}{P_r}\right) > 1 \tag{21}$$

for all $r \geq 3$, $(P_3 = 5)$

Combining this with

$$\left(\frac{P_{r+2}}{P_r}\right)\left(\frac{P_{r+2}}{P_{r+2}}\right) = \frac{P_{r+2}}{P_r} \tag{22}$$

Yeilds:

$$\left(\frac{P_{r+2}-2}{P_{r+1}}\right)\left(\frac{P_{r+1}-2}{P_r}\right)\left(\frac{P_{r+2}}{P_r}\right)\left(\frac{P_{r+2}}{P_{r+2}}\right) > \frac{P_{r+2}}{P_r} \tag{23}$$

Combining (23) with the the induction hypothesis yields:

$$\left(\frac{P_{r+2}-2}{P_{r+1}}\right)\left(\frac{P_{r+1}-2}{P_r}\right)\left(\frac{P_{r+2}}{P_r}\right)\left(\frac{P_{r+2}}{P_{r+2}}\right)\left(P_r^2\prod_{j=1}^{r}C\left(j,0\right)\right) > \left(P_r+2\right)\left(\frac{P_{r+2}}{P_r}\right) \tag{24a}$$

$$\left(\frac{P_{r+2}-2}{P_{r+1}}\right)\left(\frac{P_{r+1}-2}{P_r}\right)\left(\frac{P_{r+2}}{P_r}\right)\left(\frac{P_{r+2}}{P_{r+2}}\right)\left(P_r^2\prod_{j=1}^{r}C\left(j,0\right)\right) > P_{r+2}+2\left(\frac{P_{r+2}}{P_r}\right) \tag{24b}$$

$$\left(\frac{P_{r+2}-2}{P_{r+1}}\right)\left(\frac{P_{r+1}-2}{P_r}\right)\left(\frac{P_{r+2}}{P_r}\right)\left(\frac{P_{r+2}}{P_{r+2}}\right)\left(P_r^2\prod_{j=1}^{r}C\left(j,0\right)\right) > P_{r+2}+2 \tag{24c}$$

$$\left(\frac{P_{r+2}}{P_r}\right)\left(\frac{P_{r+2}}{P_r}\right)\left(\frac{P_{r+2}-2}{P_{r+2}}\right)\left(\frac{P_{r+1}-2}{P_{r+1}}\right)\left(P_r^2\prod_{j=1}^{r}C\left(j,0\right)\right) > P_{r+2}+2 \tag{24d}$$

$$\left(\frac{P_{r+2}^2}{P_r^2}\right)\left(P_r^2\right)\left(\frac{P_{r+2}-2}{P_{r+2}}\right)\left(\frac{P_{r+1}-2}{P_{r+1}}\right)\left(\prod_{j=1}^{r}C\left(j,0\right)\right) > P_{r+2}+2 \tag{24e}$$

Noting that $C(r+2,0) = \frac{P_{r+2}-2}{P_{r+2}}$ and $C(r+1,0) = \frac{P_{r+1}-2}{P_{r+1}}$ Equation (24e) becomes:

$$\left(P_{r+2}^2\right)\prod_{j=1}^{r+2}C\left(j,0\right) > P_{r+2}+2 \tag{25}$$

thus,

$$\left(P_r^2\right)\prod_{j=1}^{r}C\left(j,0\right) > P_r+2 \Longrightarrow \left(P_{r+2}^2\right)\prod_{j=1}^{r+2}C\left(j,0\right) > P_{r+2}+2 \tag{26}$$

The two successive values of $r = 12$ and $r = 13$ suffice to complete the proof.

For $r = 12$ $(P_{12} = 37)$

$$\begin{aligned}
\left(P_{12}^2\right)\prod_{j=1}^{12} C\left(j,0\right) &= \left(37^2\right)\left[\frac{(1)\,(1)\,(3)\,(5)\,(9)\,(11)\,(15)\,(17)\,(21)\,(27)\,(29)\,(35)}{(2)\,(3)\,(5)\,(7)\,(11)\,(13)\,(17)\,(19)\,(23)\,(29)\,(31)\,(37)}\right] \\
&= (1681)\left[\frac{382725}{13032214}\right] \\
&> 40 \\
&> P_{12} + 2
\end{aligned} \tag{27}$$

For $r = 13$ $(P_{13} = 41)$

$$\begin{aligned}
\left(P_{13}^2\right)\prod_{j=1}^{13} C\left(j,0\right) &= \left(41^2\right)\left[\frac{(1)\,(1)\,(3)\,(5)\,(9)\,(11)\,(15)\,(17)\,(21)\,(27)\,(29)\,(35)\,(39)}{(2)\,(3)\,(5)\,(7)\,(11)\,(13)\,(17)\,(19)\,(23)\,(29)\,(31)\,(37)\,(41)}\right] \\
&= (1681)\left[\frac{1148175}{41101598}\right] \\
&> 46 \\
&> P_{13} + 2
\end{aligned} \tag{28}$$

□

6. Lemma 2

Lemma 2.
$$\left(\frac{P_{r+2}-2}{P_{r+1}}\right)\left(\frac{P_{r+1}-2}{P_r}\right) > 1$$
For any $r \geq 3$

Proof. Let $P_{r+2} = P_{r+1} + k_2$ and $P_{r+1} = P_r + k_1$ for some constants: k_1 and k_2. then
$$\left(\frac{P_{r+2}-2}{P_{r+1}}\right)\left(\frac{P_{r+1}-2}{P_r}\right) = \left(\frac{P_{r+1}+k_2-2}{P_{r+1}}\right)\left(\frac{P_r+k_1-2}{P_r}\right) \tag{29}$$
For $r \geq 3$ ($P_3 = 5$) there are 3 cases to consider:

Case 1: $k_1 = 2$ and $k_2 = 4$.
$$\begin{aligned}\left(\frac{P_{r+1}+k_2-2}{P_{r+1}}\right)\left(\frac{P_r+k_1-2}{P_r}\right) &= \left(\frac{P_{r+1}+4-2}{P_{r+1}}\right)\left(\frac{P_r+2-2}{P_r}\right)\\ &= \left(\frac{P_{r+1}+2}{P_{r+1}}\right)\left(\frac{P_r}{P_r}\right)\\ &= \left(1+\frac{2}{P_{r+1}}\right)\\ &> 1\end{aligned}$$

Case 2: $k_1 = 4$ and $k_2 = 2$.
$$\begin{aligned}\left(\frac{P_{r+1}+k_2-2}{P_{r+1}}\right)\left(\frac{P_r+k_1-2}{P_r}\right) &= \left(\frac{P_{r+1}+2-2}{P_{r+1}}\right)\left(\frac{P_r+4-2}{P_r}\right)\\ &= \left(\frac{P_{r+1}}{P_{r+1}}\right)\left(\frac{P_r+2}{P_r}\right)\\ &= \left(1+\frac{2}{P_r}\right)\\ &> 1\end{aligned}$$

Case 3: $k_1 \geq 4$ and $k_2 \geq 4$
$$\begin{aligned}\left(\frac{P_{r+1}+k_2-2}{P_{r+1}}\right)\left(\frac{P_r+k_1-2}{P_r}\right) &\geq \left(\frac{P_{r+1}+4-2}{P_{r+1}}\right)\left(\frac{P_r+4-2}{P_r}\right)\\ &\geq \left(\frac{P_{r+1}+2}{P_{r+1}}\right)\left(\frac{P_r+2}{P_r}\right)\\ &\geq \left(1+\frac{2}{P_{r+1}}\right)\left(1+\frac{2}{P_r}\right)\\ &> 1\end{aligned}$$

In all three cases:

$$\left(\frac{P_{r+1}+k_2-2}{P_{r+1}}\right)\left(\frac{P_r+k_1-2}{P_r}\right)>1$$

combining this with equation (29) yields

$$\left(\frac{P_{r+2}-2}{P_{r+1}}\right)\left(\frac{P_{r+1}-2}{P_r}\right)>1$$

For all $r \geq 3$, $(P_3 = 5)$

□

7. Lemma 3

Lemma 3.

$$\left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi ix\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}-1}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right)\right|<2r \tag{30}$$

for all $r>5$ and all real $0<x$.

Proof. Define a function, $g\left(r,t\right)$, such that:

$$g\left(r,t\right)=\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi it\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right) \tag{31}$$

for all $r>5$ and all real $0<x$.

Using $g\left(r,t\right)$ equation (30) can be rewritten as:

$$\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi ix\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}-1}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right)=g\left(r,x\right)-g\left(r,0\right) \tag{32a}$$

$$\left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi ix\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}-1}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right)\right|\leq 2\left|g\left(r,t\right)\right| \tag{32b}$$

$$\left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi ix\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}-1}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right)\right|\leq 2\left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi it\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right)\right| \tag{32c}$$

From Lemma 4

$$\left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi it\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}}{2\pi i\sum\limits_{j=1}^{r}\frac{n_j}{P_j}}\right)\right|<r \tag{33}$$

for all $x>0$ and all $r>5$.

Combining this with equation (32c) yields:

$$\left| \dot{\sum_{\overrightarrow{n_r} \in \mathbb{Z}}} \left(\prod_{j=1}^{r} C\left(j, n_j\right) \right) \left(\frac{e^{2\pi i x \sum\limits_{j=1}^{r} \frac{n_j}{P_j}} - 1}{2\pi i \sum\limits_{j=1}^{r} \frac{n_j}{P_j}} \right) \right| < 2r \tag{34}$$

□

8. Lemma 4

Lemma 4.

$$\left| \dot{\sum_{\vec{n_r} \in \mathbb{Z}}} \prod_{j=1}^{r} C(j, n_j) \frac{e^{2\pi i x \sum_{j=1}^{r} \frac{n_j}{P_j}}}{2\pi i \sum_{j=1}^{r} \frac{n_j}{P_j}} \right| < r \tag{35}$$

for all $x > 0$ and all $r > 5$.

Proof. Assume

$$\left| \dot{\sum_{\vec{n_r} \in \mathbb{Z}}} \prod_{j=1}^{r} C(j, n_j) \frac{e^{2\pi i x \sum_{j=1}^{r} \frac{n_j}{P_j}}}{2\pi i \sum_{j=1}^{r} \frac{n_j}{P_j}} \right| < r \tag{36}$$

is true for for all $x > 0$ and some finite r.

Beginning with

$$\dot{\sum_{\vec{n_r} \in \mathbb{Z}}} \prod_{j=1}^{r+1} C(j, n_j) \frac{e^{2\pi i x \sum_{j=1}^{r+1} \frac{n_j}{P_j}}}{2\pi i \sum_{j=1}^{r+1} \frac{n_j}{P_j}} \tag{37}$$

and separating out the 3 cases:

Case 1: $\sum_{j=1}^{r} \frac{n_j}{P_j} = 0$ and $\frac{n_{r+1}}{P_{r+1}} \neq 0$

Case 2: $\sum_{j=1}^{r} \frac{n_j}{P_j} \neq 0$ and $\frac{n_{r+1}}{P_{r+1}} = 0$

Case 3: $\sum\limits_{j=1}^{r} \frac{n_j}{P_j} \neq 0$ and $\frac{n_{r+1}}{P_{r+1}} \neq 0$

Yields:

$$
\begin{aligned}
\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}} \left(\prod_{j=1}^{r+1} C\left(j, n_j\right)\right) \frac{e^{2\pi i x \sum\limits_{j=1}^{r+1} \frac{n_j}{P_j}}}{2\pi i \sum\limits_{j=1}^{r+1} \frac{n_j}{P_j}} = \left[\prod_{j=1}^{r} C\left(j, 0\right)\right] \left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}} C\left(r+1, n_{r+1}\right) \frac{e^{2\pi i t \frac{n_{r+1}}{P_{r+1}}}}{2\pi i \frac{n_{r+1}}{P_{r+1}}}\right] \\
+ C\left(r+1, 0\right) \left[\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}} \left(\prod_{j=1}^{r} C\left(j, n_j\right)\right) \frac{e^{2\pi i x \sum\limits_{j=1}^{r} \frac{n_j}{P_j}}}{2\pi i \sum\limits_{j=1}^{r} \frac{n_j}{P_j}}\right] \\
+ \left[\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}} \prod_{j=1}^{r} C\left(j, n_j\right)\right] \left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}} \left(C\left(r+1, n_{r+1}\right)\right) \frac{e^{2\pi i t \left(\frac{n_{r+1}}{P_{r+1}} + \sum\limits_{j=1}^{r} \frac{n_j}{P_j}\right)}}{2\pi i \left(\frac{n_{r+1}}{P_{r+1}} + \sum\limits_{j=1}^{r} \frac{n_j}{P_j}\right)}\right]
\end{aligned}
\tag{38}
$$

Substituting $\frac{s_r}{M_r}$ for $\sum\limits_{j=1}^{r} \frac{n_j}{P_j}$ equation (38) becomes:

$$\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}} \left(\prod_{j=1}^{r+1} C(j,n_j)\right) \frac{e^{2\pi i x \sum_{j=1}^{r+1} \frac{n_j}{P_j}}}{2\pi i \sum_{j=1}^{r+1} \frac{n_j}{P_j}} = \left[\prod_{j=1}^{r} C(j,0)\right] \left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}} C(r+1,n_{r+1}) \frac{e^{2\pi i t \frac{n_{r+1}}{P_{r+1}}}}{2\pi i \frac{n_{r+1}}{P_{r+1}}}\right]$$
$$+ C(r+1,0) \left[\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}} \left(\prod_{j=1}^{r} C(j,n_j)\right) \frac{e^{2\pi i x \frac{s_r}{M_r}}}{2\pi i \frac{s_r}{M_r}}\right]$$
$$+ \left[\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}} \prod_{j=1}^{r} C(j,n_j)\right] \left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}} \left(C(r+1,n_{r+1})\right) \frac{e^{2\pi i t\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i \left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}\right] \tag{39}$$

using the definition of $C(r+1,0)$, and the definition of $C(r+1,n_{r+1})$ when $n_{r+1} \neq 0$ equation (39) becomes:

$$\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\left(\prod_{j=1}^{r+1}C\left(j,n_j\right)\right)\frac{e^{2\pi ix\sum_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r+1}\frac{n_j}{P_j}}=\left[\prod_{j=1}^{r}C\left(j,0\right)\right]\left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}}C\left(r+1,n_{r+1}\right)\frac{e^{2\pi it\frac{n_{r+1}}{P_{r+1}}}}{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right]$$
$$+\left(1-\frac{2}{P_{r+1}}\right)\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}}$$
$$+\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\prod_{j=1}^{r}C\left(j,n_j\right)\dot{\sum_{n_{r+1}\in\mathbb{Z}}}\left(\frac{\left(1+e^{-4\pi i\frac{n_j}{P_{r+1}}}\right)\left(1-e^{2\pi i\frac{n_j}{P_{r+1}}}\right)}{2\pi in_{r+1}}\right)\frac{e^{2\pi it\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)} \tag{40}$$

$$\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\left(\prod_{j=1}^{r+1}C\left(j,n_j\right)\right)\frac{e^{2\pi ix\sum\limits_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum\limits_{j=1}^{r+1}\frac{n_j}{P_j}}=\left[\prod_{j=1}^{r}C\left(j,0\right)\right]\left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}}C\left(r+1,n_{r+1}\right)\frac{e^{2\pi it\frac{n_{r+1}}{P_{r+1}}}}{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right]$$
$$+\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}}$$
$$-\frac{2}{P_{r+1}}\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}}$$
$$+\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\prod_{j=1}^{r}C\left(j,n_j\right)\dot{\sum_{n_{r+1}\in\mathbb{Z}}}\left(\frac{\left(1+e^{-4\pi i\frac{n_{r+1}}{P_{r+1}}}\right)\left(1-e^{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right)}{2\pi in_{r+1}}\right)\frac{e^{2\pi it\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)} \tag{41}$$

Noting that:

$$\lim_{n_{r+1}\to 0}\left[\left(\frac{\left(1+e^{-4\pi i\frac{n_{r+1}}{P_{r+1}}}\right)\left(1-e^{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right)}{2\pi in_{r+1}}\right)\frac{e^{2\pi it\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}\right)=\left(-\frac{2}{P_{r+1}}\right)\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}}$$

equation (41) becomes:

$$\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\left(\prod_{j=1}^{r+1}C\left(j,n_j\right)\right)\frac{e^{2\pi i x\sum_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r+1}\frac{n_j}{P_j}}=\left[\prod_{j=1}^{r}C\left(j,0\right)\right]\left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}}C\left(r+1,n_{r+1}\right)\frac{e^{2\pi i t\frac{n_{r+1}}{P_{r+1}}}}{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right]$$
$$+\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\frac{e^{2\pi i t\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}}$$
$$+\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\prod_{j=1}^{r}C\left(j,n_j\right)\sum_{n_{r+1}\in\mathbb{Z}}\left(\frac{\left(1+e^{-4\pi i\frac{n_{r+1}}{P_{r+1}}}\right)\left(1-e^{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right)}{2\pi i n_{r+1}}\right)\frac{e^{2\pi i t\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)} \tag{42}$$

From Lemma 6

$$\sum_{n_{r+1}\in\mathbb{Z}}\left(\frac{\left(1+e^{-4\pi i\frac{n_{r+1}}{P_{r+1}}}\right)\left(1-e^{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right)}{2\pi i n_{r+1}}\right)\frac{e^{2\pi i t\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}=\begin{cases}-\frac{e^{2\pi i t\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}} & kP_r+1\le x<kP_r+2\\ -\frac{e^{2\pi i t\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}} & kP_r-1\le x<kP_r\\ 0 & \text{otherwise}\end{cases} \tag{43}$$

From this equation (42) becomes:

$$\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\left(\prod_{j=1}^{r+1}C\left(j,n_j\right)\right)\frac{e^{2\pi ix\sum_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r+1}\frac{n_j}{P_j}}=\left[\prod_{j=1}^{r}C\left(j,0\right)\right]\left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}}C\left(r+1,n_{r+1}\right)\frac{e^{2\pi it\frac{n_{r+1}}{P_{r+1}}}}{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right]$$
$$+\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}}$$
$$+\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left[\begin{cases}-\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}} & kP_r+1\leq x<kP_r+2\\ -\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}} & kP_r-1\leq x<kP_r\\ 0 & \text{otherwise}\end{cases}\right] \tag{44}$$

Combining the last two terms on the right hand side of equation (44) yields:

$$\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\left(\prod_{j=1}^{r+1}C\left(j,n_j\right)\right)\frac{e^{2\pi ix\sum_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r+1}\frac{n_j}{P_j}}=\left[\prod_{j=1}^{r}C\left(j,0\right)\right]\left[\dot{\sum_{n_{r+1}\in\mathbb{Z}}}C\left(r+1,n_{r+1}\right)\frac{e^{2\pi it\frac{n_{r+1}}{P_{r+1}}}}{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right]$$
$$+\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left[\begin{cases}0 & kP_r+1\leq x<kP_r+2\\ 0 & kP_r-1\leq x<kP_r\\ \frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}} & \text{otherwise}\end{cases}\right] \tag{45}$$

Applying the triangle inequality to equation (45) yeilds:

$$\left|\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\left(\prod_{j=1}^{r+1}C\left(j,n_j\right)\right)\frac{e^{2\pi ix\sum_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r+1}\frac{n_j}{P_j}}\right| \leq \prod_{j=1}^{r}C\left(j,0\right)\left|\dot{\sum_{n_{r+1}\in\mathbb{Z}}}C\left(r+1,n_{r+1}\right)\frac{e^{2\pi it\frac{n_{r+1}}{P_{r+1}}}}{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right| + \left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}}\right)\right| \tag{46}$$

From Lemma 5

$$\left|\dot{\sum_{n_r\in\mathbb{Z}}}C\left(r,n_r\right)\frac{e^{2\pi it\frac{n_r}{P_r}}}{2\pi i\frac{n_r}{P_r}}\right| \leq \left(1-\frac{3}{P_r}\right) \tag{47}$$

Combining this with the inequality (46) yields

$$\left|\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\left(\prod_{j=1}^{r+1}C\left(j,n_j\right)\right)\frac{e^{2\pi ix\sum_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r+1}\frac{n_j}{P_j}}\right| \leq \left(1-\frac{3}{P_r}\right)\prod_{j=1}^{r}C\left(j,0\right) + \left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{r}C\left(j,n_j\right)\right)\left(\frac{e^{2\pi it\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}}\right)\right| \tag{48}$$

Noting that

$$\left(1-\frac{3}{P_r}\right)\prod_{j=1}^{r}C\left(j,0\right) < 1 \tag{49}$$

and the induction hypthesis, (36), inequality (48) becomes:

$$\left|\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\left(\prod_{j=1}^{r+1}C\left(j,n_j\right)\right)\frac{e^{2\pi ix\sum_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r+1}\frac{n_j}{P_j}}\right|<1+r \tag{50}$$

Thus:

$$\left|\dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\prod_{j=1}^{r}C\left(j,n_j\right)\frac{e^{2\pi ix\sum_{j=1}^{r}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r}\frac{n_j}{P_j}}\right|<r\Longrightarrow\left|\dot{\sum_{\overrightarrow{n_{r+1}}\in\mathbb{Z}}}\prod_{j=1}^{r+1}C\left(j,n_j\right)\frac{e^{2\pi ix\sum_{j=1}^{r+1}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{r+1}\frac{n_j}{P_j}}\right|<r+1 \tag{51}$$

To complete the proof it is necessary to show the left hand side of (51) is true for some, finite r.
Appendix 9 provides a table of values for for the case of $r=5$ ($P_5=11$).
The function, $G(5,x)$ has a maximun value of $2\frac{2197}{2310}$ at $x=643$.
The function, $G(5,x)$ has a minimun value of $-2\frac{2197}{2310}$ at $x=1668$.
Thus, for $r=5$:

$$\left|\dot{\sum_{\overrightarrow{n_5}\in\mathbb{Z}}}\prod_{j=1}^{5}C\left(j,n_j\right)\frac{e^{2\pi ix\sum_{j=1}^{5}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{5}\frac{n_j}{P_j}}\right|\leq 2\frac{2197}{2310}<5$$

Thus:

$$\left|\dot{\sum_{\overrightarrow{n_5}\in\mathbb{Z}}}\prod_{j=1}^{5} C\left(j,n_j\right)\frac{e^{2\pi ix\sum_{j=1}^{5}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{5}\frac{n_j}{P_j}}\right| < r \tag{52}$$

for all $r \geq 5$.

$\square$

9. Lemma 5

Lemma 5.

$$\left|\dot{\sum_{n_r\in\mathbb{Z}}} C\left(r,n_r\right)\left(\frac{e^{2\pi it\frac{n_r}{P_r}}}{2\pi i\frac{n_r}{P_r}}\right)\right| \leq \left(1-\frac{3}{P_r}\right) \tag{53}$$

for all $x > 0$ and all $r > 2$.

Proof. Consider th function:

$$g(t,P) = \begin{cases} \left(\frac{2t-1}{P}\right) & 0 < t < 1 \\ \left(1-\frac{1}{P}\right) - t\left(1-\frac{2}{P}\right) & 1 < t < 2 \\ \frac{2t}{P} - \left(1+\frac{1}{P}\right) & 2 < t < P-1 \\ \left(P-2-\frac{1}{P}\right) - t\left(1-\frac{2}{P}\right) & P-1 < t < P \end{cases} \tag{54}$$

The Fourier constants for the funtion, $g(t,P)$ are given by:

$$c_0 = \frac{1}{P}\int_{t=0}^{P} g(t,P)dt = 0 \tag{55a}$$

$$c_n = \frac{1}{P}\int_{t=0}^{P} g(t,P)e^{-2\pi it\frac{n}{P}}dt = \frac{P\left(1-e^{-4\pi\frac{n}{P}}\right)\left(1+e^{2\pi\frac{n}{P}}\right)}{4\pi^2n^2} \tag{55b}$$

It is clear from the definition

$$C\left(r,n_r\right) = \frac{\left(1+e^{-4\pi\frac{n_r}{P_r}}\right)\left(1-e^{2\pi\frac{n_r}{P_r}}\right)}{2\pi in_r} \tag{56}$$

for all $r > 2$ that

$$\dot{\sum_{n_r\in\mathbb{Z}}} C\left(r,n_r\right)\left(\frac{e^{2\pi it\frac{n_r}{P_r}}}{2\pi i\frac{n_r}{P_r}}\right) = g(t,P_r) \tag{57}$$

and that

$$\left|\dot{\sum_{n_r\in\mathbb{Z}}} C\left(r,n_r\right)\left(\frac{e^{2\pi it\frac{n_r}{P_r}}}{2\pi i\frac{n_r}{P_r}}\right)\right| = |g(t,P_r)| \tag{58}$$

The minima and maxima of $g(t,P_r)$ occur at $\{0,1,2,P_r-1,P_r\}$ and respectively assume the values of $\left\{-\frac{1}{P_r},\frac{1}{P_r},-\left(1-\frac{3}{P_r}\right),\left(1-\frac{3}{P_r}\right),-\frac{1}{P_r}\right\}$ at those points.

From this

$$|g(t,P_r)| \leq 1-\frac{3}{P_r} \tag{59}$$

Combining this inequality with equation (58) yields:

$$\left|\sum_{n_r\in\mathbb{Z}}^{\bullet} C\left(r,n_r\right)\left(\frac{e^{2\pi it\frac{n_r}{P_r}}}{2\pi i\frac{n_r}{P_r}}\right)\right| \leq \left(1-\frac{3}{P_r}\right) \tag{60}$$

for all $x>0$ and all $r>2$.

□

10. Lemma 6

Lemma 6.

$$\sum_{n_{r+1}\in\mathbb{Z}}\left(\frac{\left(1+e^{-4\pi i\frac{n_{r+1}}{P_{r+1}}}\right)\left(1-e^{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right)}{2\pi i n_{r+1}}\right)\frac{e^{2\pi i x\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}=\begin{cases}-\frac{e^{2\pi i x\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}} & kP_{r+1}+1\le z<kP_{r+1}+2\\ -\frac{e^{2\pi i x\frac{s_r}{M_r}}}{2\pi i\frac{s_r}{M_r}} & kP_{r+1}-1\le z<kP_{r+1}\\ 0 & otherwise\end{cases}\tag{61}$$

Proof. The Fourier transform, $\hat{g}(\xi)$ of a function $g(z)$ is definied as:

$$\hat{g}(\xi)=\int_{z=\infty}^{\infty}g(z)\,e^{-2\pi i z\xi}dz\tag{62}$$

For sampling on the integers the Poisson summation formula states:

$$\sum_{n\in\mathbb{Z}}g(n)=\sum_{n\in\mathbb{Z}}\hat{g}(n)\tag{63}$$

On the the lefthand side of (61) there are two values which could pose a problem: $n_{r+1}=0$ and $n_{r+1}=-P_{r+1}\frac{s_r}{M_r}$. At $z=0$:

$$\lim_{n_{r+1}\to 0}\frac{\left(1-e^{2\pi i\frac{n_{r+1}}{P_{r+1}}}\right)}{2\pi i n_{r+1}}=-\frac{2}{P_{r+1}}$$

For the second pole, $n_{r+1}=-P_{r+1}\frac{s_r}{M_r}$, n_{r+1} is and integer and the expression, $P_{r+1}\frac{s_r}{M_r}$ is only an integer when $\frac{s_r}{M_r}=0$. The puncuated summation though specifically excludes the value: $\frac{s_r}{M_r}=0$

Because of the behavior at $n_{r+1} = 0$ and the avoidance of the other pole at $-P_{r+1}\frac{s_r}{M_r}$, the lefthand side of (61) is smooth, integrable and has a well-defined Fourier Transform of:

$$\int_{z=\infty}^{\infty} e^{-2\pi i z\xi} \left[\left(\frac{\left(1+e^{-4\pi i \frac{z}{P_{r+1}}}\right)\left(1-e^{2\pi i \frac{z}{P_{r+1}}}\right)}{2\pi i z} \right) \frac{e^{2\pi i x\left(\frac{z}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i\left(\frac{z}{P_{r+1}}+\frac{s_r}{M_r}\right)} \right] dz = \begin{cases} -\frac{e^{2\pi i x \frac{s_r}{M_r}}}{2\pi i \frac{s_r}{M_r}} & kP_{r+1}+1 \le x < kP_{r+1}+2 \\ -\frac{e^{2\pi i t \frac{s_r}{M_r}}}{2\pi i \frac{s_r}{M_r}} & kP_{r+1}-1 \le x < kP_{r+1} \\ 0 & \text{otherwise} \end{cases} \tag{64}$$

; where $k \in \mathbb{Z}$

Using the values of the Fourier transform in equation (64) with the Poisson resummation formula of (63) yields:

$$\sum_{n_{r+1}\in\mathbb{Z}} \left(\frac{\left(1+e^{-4\pi i \frac{n_{r+1}}{P_{r+1}}}\right)\left(1-e^{2\pi i \frac{n_{r+1}}{P_{r+1}}}\right)}{2\pi i n_{r+1}} \right) \frac{e^{2\pi i x\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)}}{2\pi i\left(\frac{n_{r+1}}{P_{r+1}}+\frac{s_r}{M_r}\right)} = \begin{cases} -\frac{e^{2\pi i x \frac{s_r}{M_r}}}{2\pi i \frac{s_r}{M_r}} & n_{r+1}P_{r+1}+1 \le x < n_{r+1}P_{r+1}+2 \\ -\frac{e^{2\pi i x \frac{s_r}{M_r}}}{2\pi i \frac{s_r}{M_r}} & n_{r+1}P_{r+1}-1 \le x < n_{r+1}P_{r+1} \\ 0 & \text{otherwise} \end{cases} \tag{65}$$

□

Appendices

Appendix A:
Table of Values of the 5-Stranded Sieve

The functions: $T(5,x)$, $A(5,x)$, and $V(5,x)$, are described in section 2. For the case, $r=5$, $A(5,x)$ is defined as:

$$A(5,t) = x\frac{9}{154}$$

and $V(5,x)$ is defined as:

$$V(5,t) = T(5,t) - A(5,t)$$

The function, $G(r,x)$ is defined section Sec:LemmaB as part of lemma 3. For the case, $r=5$, $G(r,x)$ is defined as:

$$G(5,t) = \dot{\sum_{\overrightarrow{n_r}\in\mathbb{Z}}}\left(\prod_{j=1}^{5} C(j,n_j)\right)\left(\frac{e^{2\pi it\sum_{j=1}^{5}\frac{n_j}{P_j}}}{2\pi i\sum_{j=1}^{5}\frac{n_j}{P_j}}\right)$$
$$= V(5,t) - \frac{145}{308}$$

which also equals $G(5,x) = V(5,x) - \frac{145}{308}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
0	0	0	0	$-\frac{1087}{2310}$
1	1	$0\frac{9}{154}$	$\frac{145}{154}$	$\frac{1087}{2310}$
18	1	$1\frac{4}{77}$	$-\frac{4}{77}$	$-\frac{1207}{2310}$
19	2	$1\frac{17}{154}$	$\frac{137}{154}$	$\frac{967}{2310}$
30	2	$1\frac{58}{77}$	$\frac{19}{77}$	$-\frac{47}{210}$
31	3	$1\frac{125}{154}$	$1\frac{29}{154}$	$1\frac{326}{1155}$
42	3	$2\frac{5}{11}$	$\frac{6}{11}$	$\frac{86}{1155}$
43	4	$2\frac{79}{154}$	$1\frac{75}{154}$	$1\frac{37}{2310}$
60	4	$3\frac{39}{77}$	$\frac{38}{77}$	$\frac{26}{1155}$
61	5	$3\frac{87}{154}$	$1\frac{67}{154}$	$1\frac{41}{1155}$
72	5	$4\frac{16}{77}$	$\frac{61}{77}$	$\frac{53}{165}$
73	6	$4\frac{41}{154}$	$1\frac{113}{154}$	$1\frac{607}{2310}$
102	6	$5\frac{74}{77}$	$\frac{3}{77}$	$-\frac{997}{2310}$
103	7	$6\frac{3}{154}$	$\frac{151}{154}$	$\frac{107}{210}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
108	7	$6\frac{24}{77}$	$\frac{53}{77}$	$\frac{251}{1155}$
109	8	$6\frac{57}{154}$	$1\frac{97}{154}$	$1\frac{367}{2310}$
138	8	$8\frac{5}{77}$	$-\frac{5}{77}$	$-\frac{1237}{2310}$
139	9	$8\frac{19}{154}$	$\frac{135}{154}$	$\frac{937}{2310}$
150	9	$8\frac{59}{77}$	$\frac{18}{77}$	$-\frac{547}{2310}$
151	10	$8\frac{127}{154}$	$1\frac{27}{154}$	$1\frac{31}{105}$
168	10	$9\frac{9}{11}$	$\frac{2}{11}$	$-\frac{667}{2310}$
169	11	$9\frac{135}{154}$	$1\frac{19}{154}$	$1\frac{401}{1155}$
180	11	$10\frac{79}{154}$	$\frac{37}{77}$	$\frac{1}{105}$
181	12	$10\frac{89}{154}$	$1\frac{32}{77}$	$1\frac{8}{165}$
192	12	$11\frac{17}{77}$	$\frac{60}{77}$	$\frac{356}{1155}$
193	13	$11\frac{43}{154}$	$1\frac{111}{154}$	$1\frac{577}{2310}$
198	13	$11\frac{4}{7}$	$1\frac{3}{7}$	$1\frac{97}{2310}$
199	14	$11\frac{48}{77}$	$2\frac{57}{154}$	$2\frac{116}{1155}$
222	14	$12\frac{75}{77}$	$1\frac{2}{77}$	$1\frac{1027}{2310}$
223	15	$13\frac{2}{77}$	$1\frac{149}{154}$	$1\frac{1147}{2310}$
228	15	$13\frac{7}{22}$	$1\frac{52}{77}$	$1\frac{236}{1155}$
229	16	$13\frac{59}{154}$	$2\frac{47}{77}$	$2\frac{337}{2310}$
240	16	$14\frac{2}{77}$	$1\frac{75}{77}$	$1\frac{83}{165}$
241	17	$14\frac{13}{154}$	$2\frac{141}{154}$	$2\frac{1027}{2310}$
270	17	$15\frac{60}{77}$	$1\frac{17}{77}$	$1\frac{577}{2310}$
271	18	$15\frac{129}{154}$	$2\frac{25}{154}$	$2\frac{356}{1155}$
282	18	$16\frac{73}{154}$	$1\frac{40}{77}$	$1\frac{8}{165}$
283	19	$16\frac{41}{77}$	$2\frac{71}{154}$	$2\frac{1}{105}$
312	19	$18\frac{5}{22}$	$\frac{59}{77}$	$\frac{31}{105}$
313	20	$18\frac{45}{154}$	$1\frac{109}{154}$	$1\frac{547}{2310}$
348	20	$20\frac{51}{154}$	$-\frac{51}{154}$	$-\frac{1867}{2310}$
349	21	$20\frac{30}{77}$	$\frac{93}{154}$	$\frac{307}{2310}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
360	21	$21\frac{5}{154}$	$-\frac{5}{154}$	$-\frac{107}{210}$
361	22	$21\frac{15}{154}$	$\frac{139}{154}$	$\frac{997}{2310}$
378	22	$22\frac{13}{154}$	$-\frac{13}{154}$	$-\frac{1297}{2310}$
379	23	$22\frac{1}{7}$	$\frac{131}{154}$	$\frac{877}{2310}$
390	23	$22\frac{61}{77}$	$\frac{31}{154}$	$-\frac{607}{2310}$
391	24	$22\frac{131}{154}$	$1\frac{1}{7}$	$1\frac{53}{165}$
402	24	$23\frac{38}{77}$	$\frac{1}{2}$	$\frac{41}{1155}$
403	25	$23\frac{85}{154}$	$1\frac{34}{77}$	$1\frac{26}{1155}$
420	25	$24\frac{6}{11}$	$\frac{69}{154}$	$-\frac{37}{2310}$
421	26	$24\frac{93}{154}$	$1\frac{30}{77}$	$1\frac{86}{1155}$
432	26	$25\frac{19}{77}$	$\frac{58}{77}$	$\frac{326}{1155}$
433	27	$25\frac{47}{154}$	$1\frac{107}{154}$	$1\frac{47}{210}$
438	27	$25\frac{46}{77}$	$1\frac{61}{154}$	$1\frac{157}{2310}$
439	28	$25\frac{101}{154}$	$2\frac{26}{77}$	$2\frac{146}{1155}$
462	28	27	1	$1\frac{1087}{2310}$
463	29	$27\frac{4}{77}$	$1\frac{145}{154}$	$1\frac{1087}{2310}$
480	29	$28\frac{4}{77}$	$\frac{73}{77}$	$\frac{551}{1155}$
481	30	$28\frac{17}{154}$	$1\frac{137}{154}$	$1\frac{967}{2310}$
492	30	$28\frac{58}{77}$	$1\frac{19}{77}$	$1\frac{47}{210}$
493	31	$28\frac{125}{154}$	$2\frac{29}{154}$	$2\frac{326}{1155}$
522	31	$30\frac{1}{2}$	$\frac{38}{77}$	$\frac{26}{1155}$
523	32	$30\frac{43}{77}$	$1\frac{67}{154}$	$1\frac{41}{1155}$
528	32	$30\frac{6}{7}$	$1\frac{3}{22}$	$1\frac{757}{2310}$
529	33	$30\frac{141}{154}$	$2\frac{13}{154}$	$2\frac{446}{1155}$
558	33	$32\frac{47}{77}$	$\frac{30}{77}$	$-\frac{17}{210}$
559	34	$32\frac{103}{154}$	$1\frac{51}{154}$	$1\frac{23}{165}$
570	34	$33\frac{24}{77}$	$\frac{53}{77}$	$\frac{251}{1155}$
571	35	$33\frac{57}{154}$	$1\frac{48}{77}$	$1\frac{367}{2310}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
588	35	$34\frac{4}{11}$	$\frac{97}{154}$	$\frac{191}{1155}$
589	36	$34\frac{65}{154}$	$1\frac{4}{7}$	$1\frac{247}{2310}$
600	36	$35\frac{9}{154}$	$\frac{72}{77}$	$\frac{536}{1155}$
601	37	$35\frac{9}{77}$	$1\frac{135}{154}$	$1\frac{937}{2310}$
612	37	$35\frac{59}{77}$	$1\frac{5}{22}$	$1\frac{547}{2310}$
613	38	$35\frac{127}{154}$	$2\frac{13}{77}$	$2\frac{31}{105}$
618	38	$36\frac{17}{154}$	$1\frac{68}{77}$	$1\frac{68}{165}$
619	39	$36\frac{13}{77}$	$2\frac{127}{154}$	$2\frac{817}{2310}$
630	39	$36\frac{9}{11}$	$2\frac{27}{154}$	$2\frac{667}{2310}$
631	40	$36\frac{135}{154}$	$3\frac{9}{77}$	$3\frac{401}{1155}$
642	40	$37\frac{79}{154}$	$2\frac{37}{77}$	$2\frac{1}{105}$
643	41	$37\frac{4}{7}$	$3\frac{65}{154}$	$3\frac{8}{165}$
660	41	$38\frac{87}{154}$	$2\frac{3}{7}$	$2\frac{97}{2310}$
661	42	$38\frac{48}{77}$	$3\frac{57}{154}$	$3\frac{116}{1155}$
690	42	$40\frac{25}{77}$	$1\frac{52}{77}$	$1\frac{236}{1155}$
691	43	$40\frac{59}{154}$	$2\frac{47}{77}$	$2\frac{337}{2310}$
702	43	$41\frac{3}{154}$	$1\frac{75}{77}$	$1\frac{83}{165}$
703	44	$41\frac{13}{154}$	$2\frac{10}{11}$	$2\frac{1027}{2310}$
732	44	$42\frac{60}{77}$	$1\frac{17}{77}$	$1\frac{577}{2310}$
733	45	$42\frac{129}{154}$	$2\frac{25}{154}$	$2\frac{356}{1155}$
768	45	$44\frac{68}{77}$	$\frac{17}{154}$	$-\frac{817}{2310}$
769	46	$44\frac{145}{154}$	$1\frac{4}{77}$	$1\frac{68}{165}$
798	46	$46\frac{97}{154}$	$-\frac{97}{154}$	$-\frac{2557}{2310}$
799	47	$46\frac{107}{154}$	$\frac{23}{77}$	$-\frac{191}{1155}$
810	47	$47\frac{51}{154}$	$-\frac{51}{154}$	$-\frac{1867}{2310}$
811	48	$47\frac{30}{77}$	$\frac{93}{154}$	$\frac{307}{2310}$
822	48	$48\frac{3}{77}$	$-\frac{3}{77}$	$-\frac{107}{210}$
823	49	$48\frac{15}{154}$	$\frac{139}{154}$	$\frac{997}{2310}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
828	49	$48\frac{30}{77}$	$\frac{47}{77}$	$\frac{23}{165}$
829	50	$48\frac{34}{77}$	$1\frac{85}{154}$	$1\frac{17}{210}$
840	50	$49\frac{1}{11}$	$\frac{10}{11}$	$\frac{46}{105}$
841	51	$49\frac{23}{154}$	$1\frac{131}{154}$	$1\frac{877}{2310}$
852	51	$49\frac{61}{77}$	$1\frac{16}{77}$	$1\frac{607}{2310}$
853	52	$49\frac{131}{154}$	$2\frac{23}{154}$	$2\frac{53}{165}$
858	52	$50\frac{1}{7}$	$1\frac{6}{7}$	$1\frac{446}{1155}$
859	53	$50\frac{31}{154}$	$2\frac{123}{154}$	$2\frac{757}{2310}$
882	53	$51\frac{6}{11}$	$1\frac{69}{154}$	$1\frac{37}{2310}$
883	54	$51\frac{93}{154}$	$2\frac{30}{77}$	$2\frac{86}{1155}$
900	54	$52\frac{46}{77}$	$1\frac{61}{154}$	$1\frac{157}{2310}$
901	55	$52\frac{101}{154}$	$2\frac{26}{77}$	$2\frac{146}{1155}$
942	55	$55\frac{4}{77}$	$-\frac{4}{77}$	$-\frac{1207}{2310}$
943	56	$55\frac{17}{154}$	$\frac{137}{154}$	$\frac{967}{2310}$
948	56	$55\frac{61}{154}$	$\frac{46}{77}$	$\frac{146}{1155}$
949	57	$55\frac{5}{11}$	$1\frac{83}{154}$	$1\frac{157}{2310}$
990	57	$57\frac{6}{7}$	$-\frac{6}{7}$	$-\frac{3067}{2310}$
991	58	$57\frac{10}{11}$	$\frac{13}{154}$	$-\frac{446}{1155}$
1008	58	$58\frac{10}{11}$	$-\frac{10}{11}$	$-\frac{3187}{2310}$
1009	59	$58\frac{149}{154}$	$\frac{5}{154}$	$-\frac{46}{105}$
1020	59	$59\frac{47}{77}$	$-\frac{47}{77}$	$-\frac{227}{210}$
1021	60	$59\frac{103}{154}$	$\frac{51}{154}$	$-\frac{23}{165}$
1032	60	$60\frac{24}{77}$	$-\frac{24}{77}$	$-\frac{1807}{2310}$
1033	61	$60\frac{57}{154}$	$\frac{48}{77}$	$\frac{367}{2310}$
1038	61	$60\frac{51}{77}$	$\frac{51}{154}$	$-\frac{307}{2310}$
1039	62	$60\frac{111}{154}$	$1\frac{43}{154}$	$1\frac{221}{1155}$
1050	62	$61\frac{4}{11}$	$\frac{97}{154}$	$\frac{191}{1155}$
1051	63	$61\frac{65}{154}$	$1\frac{4}{7}$	$1\frac{247}{2310}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
1062	63	$62\frac{9}{154}$	$\frac{72}{77}$	$\frac{536}{1155}$
1063	64	$62\frac{9}{77}$	$1\frac{135}{154}$	$1\frac{937}{2310}$
1080	64	$63\frac{17}{154}$	$\frac{68}{77}$	$\frac{68}{165}$
1081	65	$63\frac{13}{77}$	$1\frac{127}{154}$	$1\frac{817}{2310}$
1092	65	$63\frac{9}{11}$	$1\frac{27}{154}$	$1\frac{667}{2310}$
1093	66	$63\frac{135}{154}$	$2\frac{9}{77}$	$2\frac{401}{1155}$
1122	66	$65\frac{87}{154}$	$\frac{3}{7}$	$-\frac{97}{2310}$
1123	67	$65\frac{48}{77}$	$1\frac{57}{154}$	$1\frac{116}{1155}$
1152	67	$67\frac{25}{77}$	$-\frac{25}{77}$	$-\frac{167}{210}$
1153	68	$67\frac{59}{154}$	$\frac{47}{77}$	$\frac{337}{2310}$
1158	68	$67\frac{52}{77}$	$\frac{25}{77}$	$-\frac{337}{2310}$
1159	69	$67\frac{113}{154}$	$1\frac{41}{154}$	$1\frac{236}{1155}$
1188	69	$69\frac{3}{7}$	$-\frac{3}{7}$	$-\frac{2077}{2310}$
1189	70	$69\frac{75}{154}$	$\frac{39}{77}$	$\frac{97}{2310}$
1218	70	$71\frac{2}{11}$	$-1\frac{2}{11}$	$-1\frac{137}{210}$
1219	71	$71\frac{37}{154}$	$-\frac{37}{154}$	$-\frac{821}{1155}$
1230	71	$71\frac{68}{77}$	$-\frac{68}{77}$	$-\frac{3127}{2310}$
1231	72	$71\frac{145}{154}$	$\frac{4}{77}$	$-\frac{68}{165}$
1248	72	$72\frac{13}{14}$	$-\frac{13}{14}$	$-\frac{3247}{2310}$
1249	73	$72\frac{76}{77}$	$\frac{1}{154}$	$-\frac{536}{1155}$
1260	73	$73\frac{7}{11}$	$-\frac{7}{11}$	$-\frac{2557}{2310}$
1261	74	$73\frac{107}{154}$	$\frac{23}{77}$	$-\frac{191}{1155}$
1272	74	$74\frac{51}{154}$	$-\frac{51}{154}$	$-\frac{1867}{2310}$
1273	75	$74\frac{30}{77}$	$\frac{93}{154}$	$\frac{307}{2310}$
1278	75	$74\frac{53}{77}$	$\frac{24}{77}$	$-\frac{367}{2310}$
1279	76	$74\frac{115}{154}$	$1\frac{39}{154}$	$1\frac{251}{1155}$
1290	76	$75\frac{30}{77}$	$\frac{93}{154}$	$\frac{23}{165}$
1291	77	$75\frac{69}{154}$	$1\frac{6}{11}$	$1\frac{17}{210}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
1302	77	$76\frac{1}{11}$	$\frac{10}{11}$	$\frac{46}{105}$
1303	78	$76\frac{23}{154}$	$1\frac{131}{154}$	$1\frac{877}{2310}$
1320	78	$77\frac{3}{22}$	$\frac{6}{7}$	$\frac{446}{1155}$
1321	79	$77\frac{15}{77}$	$1\frac{123}{154}$	$1\frac{757}{2310}$
1362	79	$79\frac{13}{22}$	$-\frac{13}{22}$	$-\frac{2467}{2310}$
1363	80	$79\frac{50}{77}$	$\frac{53}{154}$	$-\frac{146}{1155}$
1368	80	$79\frac{73}{77}$	$\frac{1}{22}$	$-\frac{967}{2310}$
1369	81	$80\frac{1}{154}$	$\frac{76}{77}$	$\frac{1207}{2310}$
1410	81	$82\frac{31}{77}$	$-1\frac{31}{77}$	$-1\frac{2017}{2310}$
1411	82	$82\frac{71}{154}$	$-\frac{71}{154}$	$-\frac{1076}{1155}$
1428	82	$83\frac{69}{154}$	$-1\frac{69}{154}$	$-1\frac{2137}{2310}$
1429	83	$83\frac{39}{77}$	$-\frac{39}{77}$	$-\frac{1136}{1155}$
1452	83	$84\frac{6}{7}$	$-1\frac{6}{7}$	$-1\frac{3067}{2310}$
1453	84	$84\frac{141}{154}$	$-\frac{141}{154}$	$-\frac{1601}{1155}$
1458	84	$85\frac{16}{77}$	$-1\frac{16}{77}$	$-1\frac{1567}{2310}$
1459	85	$85\frac{41}{154}$	$-\frac{41}{154}$	$-\frac{851}{1155}$
1470	85	$85\frac{10}{11}$	$-\frac{10}{11}$	$-\frac{3187}{2310}$
1471	86	$85\frac{149}{154}$	$\frac{5}{154}$	$-\frac{46}{105}$
1482	86	$86\frac{93}{154}$	$-\frac{93}{154}$	$-\frac{227}{210}$
1483	87	$86\frac{103}{154}$	$\frac{25}{77}$	$-\frac{23}{165}$
1488	87	$86\frac{74}{77}$	$\frac{5}{154}$	$-\frac{997}{2310}$
1489	88	$87\frac{3}{154}$	$\frac{75}{77}$	$\frac{107}{210}$
1500	88	$87\frac{51}{77}$	$\frac{51}{154}$	$-\frac{307}{2310}$
1501	89	$87\frac{111}{154}$	$1\frac{43}{154}$	$1\frac{221}{1155}$
1512	89	$88\frac{5}{14}$	$\frac{7}{11}$	$\frac{191}{1155}$
1513	90	$88\frac{32}{77}$	$1\frac{89}{154}$	$1\frac{247}{2310}$
1542	90	$90\frac{17}{154}$	$-\frac{17}{154}$	$-\frac{1357}{2310}$
1543	91	$90\frac{13}{77}$	$\frac{127}{154}$	$\frac{817}{2310}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
1578	91	$92\frac{17}{77}$	$-1\frac{17}{77}$	$-1\frac{1597}{2310}$
1579	92	$92\frac{43}{154}$	$-\frac{43}{154}$	$-\frac{866}{1155}$
1608	92	$93\frac{75}{77}$	$-1\frac{75}{77}$	$-1\frac{3337}{2310}$
1609	93	$94\frac{5}{154}$	$-1\frac{5}{154}$	$-1\frac{83}{165}$
1620	93	$94\frac{52}{77}$	$-1\frac{52}{77}$	$-1\frac{2647}{2310}$
1621	94	$94\frac{113}{154}$	$-\frac{113}{154}$	$-\frac{1391}{1155}$
1650	94	$96\frac{3}{7}$	$-2\frac{3}{7}$	$-2\frac{2077}{2310}$
1651	95	$96\frac{75}{154}$	$-1\frac{75}{154}$	$-1\frac{158}{165}$
1668	95	$97\frac{73}{154}$	$-2\frac{73}{154}$	$-2\frac{2197}{2310}$
1669	96	$97\frac{41}{77}$	$-1\frac{41}{77}$	$-1\frac{106}{105}$
1680	96	$98\frac{2}{11}$	$-2\frac{2}{11}$	$-2\frac{137}{210}$
1681	97	$98\frac{37}{154}$	$-1\frac{37}{154}$	$-1\frac{821}{1155}$
1692	97	$98\frac{68}{77}$	$-1\frac{68}{77}$	$-1\frac{3127}{2310}$
1693	98	$98\frac{145}{154}$	$-\frac{145}{154}$	$-\frac{233}{165}$
1698	98	$99\frac{5}{22}$	$-1\frac{5}{22}$	$-1\frac{1627}{2310}$
1699	99	$99\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{881}{1155}$
1710	99	$99\frac{13}{14}$	$-\frac{13}{14}$	$-\frac{3247}{2310}$
1711	100	$99\frac{76}{77}$	$\frac{1}{154}$	$-\frac{536}{1155}$
1722	100	$100\frac{7}{11}$	$-\frac{7}{11}$	$-\frac{2557}{2310}$
1723	101	$100\frac{107}{154}$	$\frac{23}{77}$	$-\frac{191}{1155}$
1740	101	$101\frac{53}{77}$	$-\frac{53}{77}$	$-\frac{2677}{2310}$
1741	102	$101\frac{115}{154}$	$\frac{39}{154}$	$-\frac{251}{1155}$
1752	102	$102\frac{30}{77}$	$-\frac{30}{77}$	$-\frac{1987}{2310}$
1753	103	$102\frac{69}{154}$	$\frac{6}{11}$	$\frac{17}{210}$
1782	103	$104\frac{3}{22}$	$-1\frac{3}{22}$	$-1\frac{1417}{2310}$
1783	104	$104\frac{15}{77}$	$-\frac{15}{77}$	$-\frac{776}{1155}$
1788	104	$104\frac{75}{154}$	$-\frac{75}{154}$	$-\frac{2227}{2310}$
1789	105	$104\frac{6}{11}$	$\frac{69}{154}$	$-\frac{26}{1155}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
1818	105	$106\frac{37}{154}$	$-1\frac{37}{154}$	$-1\frac{1657}{2310}$
1819	106	$106\frac{23}{77}$	$-\frac{23}{77}$	$-\frac{128}{165}$
1830	106	$106\frac{73}{77}$	$-\frac{73}{77}$	$-\frac{3277}{2310}$
1831	107	$107\frac{1}{154}$	$-\frac{1}{154}$	$-\frac{551}{1155}$
1848	107	108	-1	$-1\frac{1087}{2310}$
1849	108	$108\frac{4}{77}$	$-\frac{4}{77}$	$-\frac{611}{1155}$
1872	108	$109\frac{31}{77}$	$-1\frac{31}{77}$	$-1\frac{2017}{2310}$
1873	109	$109\frac{71}{154}$	$-\frac{71}{154}$	$-\frac{1076}{1155}$
1878	109	$109\frac{58}{77}$	$-\frac{58}{77}$	$-\frac{257}{210}$
1879	110	$109\frac{125}{154}$	$\frac{2}{11}$	$-\frac{326}{1155}$
1890	110	$110\frac{69}{154}$	$-\frac{69}{154}$	$-\frac{2137}{2310}$
1891	111	$110\frac{39}{77}$	$\frac{75}{154}$	$\frac{37}{2310}$
1908	111	$111\frac{39}{77}$	$-\frac{39}{77}$	$-\frac{2257}{2310}$
1909	112	$111\frac{87}{154}$	$\frac{3}{7}$	$-\frac{41}{1155}$
1920	112	$112\frac{16}{77}$	$-\frac{16}{77}$	$-\frac{1567}{2310}$
1921	113	$112\frac{41}{154}$	$\frac{113}{154}$	$\frac{607}{2310}$
1932	113	$112\frac{10}{11}$	$\frac{1}{11}$	$-\frac{877}{2310}$
1933	114	$112\frac{149}{154}$	$1\frac{5}{154}$	$1\frac{46}{105}$
1950	114	$113\frac{74}{77}$	$\frac{5}{154}$	$-\frac{997}{2310}$
1951	115	$114\frac{3}{154}$	$\frac{75}{77}$	$\frac{107}{210}$
1962	115	$114\frac{51}{77}$	$\frac{51}{154}$	$-\frac{307}{2310}$
1963	116	$114\frac{111}{154}$	$1\frac{43}{154}$	$1\frac{221}{1155}$
1998	116	$116\frac{59}{77}$	$-\frac{59}{77}$	$-\frac{2857}{2310}$
1999	117	$116\frac{127}{154}$	$\frac{13}{77}$	$-\frac{31}{105}$
2028	117	$118\frac{40}{77}$	$-1\frac{40}{77}$	$-1\frac{2287}{2310}$
2029	118	$118\frac{89}{154}$	$-\frac{89}{154}$	$-\frac{173}{165}$
2040	118	$119\frac{17}{77}$	$-1\frac{17}{77}$	$-1\frac{1597}{2310}$
2041	119	$119\frac{43}{154}$	$-\frac{43}{154}$	$-\frac{866}{1155}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
2070	119	$120\frac{75}{77}$	$-1\frac{75}{77}$	$-1\frac{3337}{2310}$
2071	120	$121\frac{5}{154}$	$-1\frac{5}{154}$	$-1\frac{83}{165}$
2082	120	$121\frac{52}{77}$	$-1\frac{52}{77}$	$-1\frac{2647}{2310}$
2083	121	$121\frac{113}{154}$	$-\frac{113}{154}$	$-\frac{1391}{1155}$
2088	121	$122\frac{3}{154}$	$-1\frac{3}{154}$	$-1\frac{1147}{2310}$
2089	122	$122\frac{6}{77}$	$-\frac{6}{77}$	$-\frac{641}{1155}$
2112	122	$123\frac{3}{7}$	$-1\frac{3}{7}$	$-1\frac{2077}{2310}$
2113	123	$123\frac{75}{154}$	$-\frac{75}{154}$	$-\frac{158}{165}$
2118	123	$123\frac{60}{77}$	$-\frac{60}{77}$	$-\frac{2887}{2310}$
2119	124	$123\frac{129}{154}$	$\frac{25}{154}$	$-\frac{356}{1155}$
2130	124	$124\frac{73}{154}$	$-\frac{73}{154}$	$-\frac{2197}{2310}$
2131	125	$124\frac{41}{77}$	$\frac{71}{154}$	$-\frac{1}{105}$
2142	125	$125\frac{2}{11}$	$-\frac{2}{11}$	$-\frac{137}{210}$
2143	126	$125\frac{37}{154}$	$\frac{58}{77}$	$\frac{667}{2310}$
2160	126	$126\frac{5}{22}$	$-\frac{5}{22}$	$-\frac{1627}{2310}$
2161	127	$126\frac{2}{7}$	$\frac{109}{154}$	$\frac{547}{2310}$
2172	127	$126\frac{13}{14}$	$\frac{5}{77}$	$-\frac{937}{2310}$
2173	128	$126\frac{76}{77}$	$1\frac{1}{154}$	$1\frac{536}{1155}$
2202	128	$128\frac{53}{77}$	$-\frac{53}{77}$	$-\frac{2677}{2310}$
2203	129	$128\frac{115}{154}$	$\frac{19}{77}$	$-\frac{251}{1155}$
2208	129	$129\frac{3}{77}$	$-\frac{3}{77}$	$-\frac{107}{210}$
2209	130	$129\frac{15}{154}$	$\frac{69}{77}$	$\frac{997}{2310}$
2238	130	$130\frac{61}{77}$	$-\frac{61}{77}$	$-\frac{2917}{2310}$
2239	131	$130\frac{131}{154}$	$\frac{23}{154}$	$-\frac{53}{165}$
2250	131	$131\frac{75}{154}$	$-\frac{75}{154}$	$-\frac{2227}{2310}$
2251	132	$131\frac{6}{11}$	$\frac{69}{154}$	$-\frac{26}{1155}$
2268	132	$132\frac{83}{154}$	$-\frac{83}{154}$	$-\frac{2347}{2310}$
2269	133	$132\frac{46}{77}$	$\frac{61}{154}$	$-\frac{86}{1155}$

x	$T(5,x)$	$A(5,x)$	$V(5,x)$	$G(5,x)$
2280	133	$133\frac{19}{77}$	$-\frac{19}{77}$	$-\frac{1657}{2310}$
2281	134	$133\frac{47}{154}$	$\frac{53}{77}$	$\frac{47}{210}$
2292	134	$133\frac{73}{77}$	$\frac{1}{22}$	$-\frac{967}{2310}$
2293	135	$134\frac{1}{154}$	$\frac{76}{77}$	$\frac{1207}{2310}$
2310	135	135	0	$-\frac{1087}{2310}$

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