

TABLE OF UNUSUAL FOURIER SERIES

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ABSTRACT. A simple compendium of Fourier series I have run across in my research. Most functions are $\mathbb{R} \rightarrow \mathbb{C}$. This is because the function, $f(t)$, was chosen to create a particular set of Fourier coefficients; Fourier coefficients that are rational polynomials in n of the form: $\frac{Q(n)}{P(n)}$.

The organization of the functions is by the Fourier coefficients; most of which are rational polynomials in n . The organization of the rational polynomials in n follows the organization of polynomials found in A Table of Series and Products by Eldon R. Hansen.

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1. INTRODUCTION

The organization of table follows the organization of A Table of Series and Products[1] by Eldon R. Hansen. For a periodic function, $f(t)$, with a period of, P , the Fourier series representation will generally be of the form:

$$\begin{aligned} f(t) &= \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n \frac{t}{P}} \\ &= \sum_{n \in \mathbb{Z}} \frac{Q(n)}{P(n)} e^{2\pi i n \frac{t}{P}} \end{aligned}$$

Where $Q(n)$ is a polynomial of degree s and $P(n)$ is a polynomial of degree r . The organization of the Fourier coefficients, c_n , which are rational polynomials will follow the organizational example found in chapter two, *Series Involving Rational, Factorial, and Power Functions*.

A benefit of this organizational approach is the A Table of Series and Products[1] by Eldon R. Hansen can be used to double check the functions, $f(t)$ found in this table. By using the fact that:

$$\frac{f(0^+) + f(0^-)}{2} = \frac{f(0^+) + f(P^-)}{2} = \sum_{n \in \mathbb{Z}} \frac{Q(n)}{P(n)}$$

one can lookup the corresponding series in the Hansen Table and verify the function, $f(t)$, cited in this table agrees with the more authoritative value found in the Hansen table.

For example, take the Fourier series over the range $0 < t < P$:

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n + \alpha)} e^{2\pi i n \frac{t}{P}} = \frac{\pi e^{2\pi i \alpha (\frac{1}{2} - \frac{t}{P})}}{\sin(\pi \alpha)}$$

Taken at the limits, 0^+ and P^- yields:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \frac{1}{(n + \alpha)} &= \left[\frac{\pi}{\sin(\pi \alpha)} \right] \left[\frac{e^{\pi i \alpha} + e^{-\pi i \alpha}}{2} \right] \\ &= \frac{\pi \cos(\pi \alpha)}{\sin(\pi \alpha)} \\ &= \pi \cot(\pi \alpha) \end{aligned}$$

which agrees with the Hansen table entry: ????.

Unless otherwise noted, the parameters of the functions cited in this table are subject to the following conditions.

$$\begin{aligned}
 P, t, x, \alpha, \beta, \gamma &\in \mathbb{R} \\
 \alpha, \beta, \gamma &\notin \mathbb{Z} \\
 \alpha, \beta, \gamma &\neq 0 \\
 \alpha, \beta, \gamma &\text{ are distinct} \\
 P &> 0
 \end{aligned}$$

Unless otherwise noted, the summation is over all integers or over all integers except zero. These summations over a variable n will be noted as follows:

$$\begin{aligned}
 \sum_{n \in \mathbb{Z}} &= \sum_n \\
 & , \\
 \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} &= \sum'_n
 \end{aligned}$$

For a real variable, x , $[x]$ is the floor of x and $\{x\}$ is the fractional part of x . Thus, for all real x ,

$$x = [x] + \{x\} \quad \text{where: } 0 \leq \{x\} < 1$$

Functions with complicated rational, Fourier coefficients were constructed from functions with simpler rational, Fourier coefficients using partial fractions. For example the function with Fourier coefficients of $\frac{1}{n(n+\alpha)}$ was constructed by combining the function with Fourier coefficients of $\frac{1}{n}$ with the the function with Fourier coefficients of $\frac{1}{(n+\alpha)}$. Because of the extensive use of partial fractions, an appendix of partial fraction expansions is included.

2. R=1,S=0

$$(1) \quad \sum_n \frac{1}{(n+\alpha)} e^{2\pi i n \frac{t}{P}} = \frac{\pi e^{2\pi i \alpha (\frac{1}{2} - \frac{t}{P})}}{\sin(\pi \alpha)}$$

where: $0 < t < P$

$$(2) \quad \sum_n \frac{(-1)^n}{(n+\alpha)} e^{2\pi i n \frac{t}{P}} = \frac{\pi}{\sin(\pi \alpha)} e^{-2\pi i \alpha \frac{t}{P}}$$

where: $\frac{-P}{2} < t < \frac{P}{2}$

$$(3) \quad \sum'_n \frac{1}{n} e^{2\pi i n \frac{t}{P}} = (2\pi i) \left(\frac{1}{2} - \frac{t}{P} \right)$$

where: $0 < t < P$

$$(4) \quad \sum'_n \frac{(-1)^n}{n} e^{2\pi i n \frac{t}{P}} = -2\pi i \frac{t}{P}$$

where: $\frac{-P}{2} < t < \frac{P}{2}$

3. R=2, S=0

$$(5) \quad \sum'_n \frac{1}{n^2} e^{2\pi i n \frac{t}{P}} = 2\pi^2 \left[\left(\frac{t}{P} \right)^2 - \frac{t}{P} + \frac{1}{6} \right]$$

where: $0 < t < P$

$$(6) \quad \sum'_n \frac{(-1)^n}{n^2} e^{2\pi i n \frac{t}{P}} = 2\pi^2 \left[\left(\frac{t}{P} \right)^2 - \frac{t}{12} \right]$$

where: $\frac{-P}{2} < t < \frac{P}{2}$

$$(7) \quad \sum'_n \frac{1}{n(n+\alpha)} e^{2\pi i n \frac{t}{P}} = \left[\frac{1}{\alpha^2} + \frac{2\pi i}{\alpha} \left(\frac{1}{2} - \frac{t}{P} \right) - \frac{\pi e^{2\pi i \alpha (\frac{1}{2} - \frac{t}{P})}}{\alpha \sin(\pi \alpha)} \right]$$

where: $0 < t < P$

$$(8) \quad \sum'_n \frac{(-1)^n}{n(n+\alpha)} e^{2\pi i n \frac{t}{P}} = \left[\frac{1}{\alpha^2} + \frac{(2\pi i)t}{P\alpha} - \frac{\pi e^{-2\pi i \alpha \frac{t}{P}}}{\alpha \sin(\pi \alpha)} \right]$$

where: $\frac{-P}{2} < t < \frac{P}{2}$

$$(9) \quad \sum'_n \frac{e^{2\pi i n \frac{t}{P}}}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\beta-\alpha)} \left[\frac{e^{2\pi i \alpha (\frac{1}{2} - \frac{t}{P})}}{\sin(\pi \alpha)} - \frac{e^{2\pi i \beta (\frac{1}{2} - \frac{t}{P})}}{\sin(\pi \beta)} \right]$$

where: $0 < t < P$

$$(10) \quad \sum'_n \frac{(-1)^n e^{2\pi i n \frac{t}{P}}}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\beta-\alpha)} \left[\frac{e^{-2\pi i \alpha \frac{t}{P}}}{\sin(\pi \alpha)} - \frac{e^{-2\pi i \beta \frac{t}{P}}}{\sin(\pi \beta)} \right]$$

where: $\frac{-P}{2} < t < \frac{P}{2}$

$$(11) \quad \sum'_n \frac{1}{(n+\alpha)^2} e^{2\pi i n \frac{t}{P}} = \left[1 - \frac{(2i)t \sin(\pi \alpha)}{P} e^{-\pi i \alpha} \right] \frac{\pi^2 e^{\frac{2\pi i \alpha t}{P}}}{\sin^2(\pi \alpha)}$$

where: $0 < t < P$

$$(12) \quad \sum'_n \frac{1}{(n-\alpha)^2} e^{2\pi i n \frac{t}{P}} = \left[1 + \frac{(2i)t \sin(\pi \alpha)}{P} e^{\pi i \alpha} \right] \frac{\pi^2 e^{-\frac{2\pi i \alpha t}{P}}}{\sin^2(\pi \alpha)}$$

where: $0 < t < P$

$$(13) \quad \sum_n \frac{(-1)^n}{(n+\alpha)^2} e^{2\pi i n \frac{t}{P}} = \left[\frac{(2i)t \sin(\pi\alpha)}{P} + \cos(\pi\alpha) \right] \frac{\pi^2 e^{-\frac{2\pi i \alpha t}{P}}}{\sin^2(\pi\alpha)}$$

where: $-\frac{P}{2} < t < \frac{P}{2}$

$$(14) \quad \sum_n \frac{n e^{2\pi i n \frac{t}{P}}}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\alpha-\beta)} \left[\frac{\alpha e^{2\pi i \alpha (\frac{1}{2} - \frac{t}{P})}}{\sin(\pi\alpha)} - \frac{\beta e^{2\pi i \beta (\frac{1}{2} - \frac{t}{P})}}{\sin(\pi\beta)} \right]$$

where: $0 < t < P$

$$(15) \quad \sum_n \frac{(-1)^n n e^{2\pi i n \frac{t}{P}}}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\alpha-\beta)} \left[\frac{\alpha e^{-2\pi i \alpha \frac{t}{P}}}{\sin(\pi\alpha)} - \frac{\beta e^{-2\pi i \beta \frac{t}{P}}}{\sin(\pi\beta)} \right]$$

where: $-\frac{P}{2} < t < \frac{P}{2}$

$$(16) \quad \checkmark \quad \sum_n \frac{1}{(n^2 - \alpha^2)} e^{2\pi i n \frac{t}{P}} = \frac{-\pi \cos(2\pi\alpha(\frac{1}{2} - \frac{t}{P}))}{\alpha \sin(\pi\alpha)}$$

where: $0 < t < P$

$$(17) \quad \checkmark \quad \sum_n \frac{(-1)^n}{(n^2 - \alpha^2)} e^{2\pi i n \frac{t}{P}} = \frac{-\pi \cos(2\pi\alpha \frac{t}{P})}{\alpha \sin(\pi\alpha)}$$

where: $-\frac{P}{2} < t < \frac{P}{2}$

$$(18) \quad \checkmark \quad \sum_n \frac{1}{(n^2 + \alpha^2)} e^{2\pi i n \frac{t}{P}} = \frac{\pi \cosh(2\pi\alpha(\frac{1}{2} - \frac{t}{P}))}{\alpha \sinh(\pi\alpha)}$$

where: $0 < t < P$

$$(19) \quad \checkmark \quad \sum_n \frac{(-1)^n}{(n^2 + \alpha^2)} e^{2\pi i n \frac{t}{P}} = \frac{\pi \cosh(2\pi\alpha \frac{t}{P})}{\alpha \sinh(\pi\alpha)}$$

where: $-\frac{P}{2} < t < \frac{P}{2}$

4. R=2,S=1

$$(20) \quad \sum_n \frac{ne^{2\pi in \frac{t}{P}}}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\alpha-\beta)} \left[\frac{\alpha e^{2\pi i\alpha(\frac{1}{2}-\frac{t}{P})}}{\sin(\pi\alpha)} - \frac{\beta e^{2\pi i\beta(\frac{1}{2}-\frac{t}{P})}}{\sin(\pi\beta)} \right]$$

where: $0 < t < P$

$$(21) \quad \sum_n \frac{n(-1)^n e^{2\pi in \frac{t}{P}}}{(n+\alpha)(n+\beta)} = \frac{\pi}{(\alpha-\beta)} \left[\frac{\alpha e^{-2\pi i\alpha \frac{t}{P}}}{\sin(\pi\alpha)} - \frac{\beta e^{-2\pi i\beta \frac{t}{P}}}{\sin(\pi\beta)} \right]$$

where: $\frac{-P}{2} < t < \frac{P}{2}$

$$(22) \quad ??? \sum_n \frac{n}{(n^2-\alpha^2)} e^{2\pi in \frac{t}{P}} = \frac{\pi i \sin(2\pi\alpha(\frac{1}{2}-\frac{t}{P}))}{2 \sin(\pi\alpha)}$$

where: $\frac{-P}{2} < t < \frac{P}{2}$

$$(23) \quad ??? \sum_n \frac{(-1)^n n}{(n^2-\alpha^2)} e^{2\pi in \frac{t}{P}} = \frac{\pi i \sin(2\pi\alpha \frac{t}{P})}{2 \sin(\pi\alpha)}$$

where: $0 < t < P$

$$(24) \quad ??? \sum_n \frac{n}{(n^2+\alpha^2)} e^{2\pi in \frac{t}{P}} = \frac{\pi i \sinh(2\pi\alpha(\frac{1}{2}-\frac{t}{P}))}{2 \sinh(\pi\alpha)}$$

where: $\frac{-P}{2} < t < \frac{P}{2}$

$$(25) \quad ??? \sum_n \frac{(-1)^n n}{(n^2+\alpha^2)} e^{2\pi in \frac{t}{P}} = \frac{\pi i \sinh(2\pi\alpha \frac{t}{P})}{2 \sinh(\pi\alpha)}$$

where: $0 < t < P$

To do:

$$\frac{n}{(n+\alpha)^2}$$

5. R=3,S=0

To do:

$$\frac{1}{n^3}$$

$$\frac{1}{n^2(n+\alpha)}$$

$$\frac{1}{n(n+\alpha)^2}$$

$$\frac{1}{(n+\alpha)^3}$$

$$\frac{1}{(n+\alpha)(n+\beta)^2}$$

$$\frac{1}{n(n+\alpha)(n+\beta)}$$

$$\frac{1}{(n+\alpha)(n+\beta)(n+\gamma)}$$

$$\frac{1}{n(n^2-\alpha^2)}$$

$$\frac{1}{n(n^2+\alpha^2)}$$

$$\frac{1}{(n+\alpha)(n^2-\beta^2)}$$

$$\frac{1}{(n+\alpha)(n^2+\beta^2)}$$

$$\frac{1}{(n^3+\alpha^3)}$$

6. $R=3, S=1$

To do:

$$\begin{aligned}
& \frac{n}{(n+\alpha)^3} \\
& \frac{n}{(n+\alpha)(n+\beta)^2} \\
& \frac{n}{(n+\alpha)(n+\beta)(n+\gamma)} \\
& \frac{n}{n(n^2-\alpha^2)} \\
& \frac{n}{n(n^2+\alpha^2)} \\
& \frac{n}{(n+\alpha)(n^2-\beta^2)} \\
& \frac{n}{(n+\alpha)(n^2+\beta^2)} \\
& \frac{n}{(n^3+\alpha^3)}
\end{aligned}$$

7. R=3,S=2

To do:

$$\frac{n^2}{(n + \alpha)^3}$$

$$\frac{n^2}{(n + \alpha)(n + \beta)^2}$$

$$\frac{n^2}{(n + \alpha)(n + \beta)(n + \gamma)}$$

$$\frac{n^2}{n(n^2 - \alpha^2)}$$

$$\frac{n^2}{n(n^2 + \alpha^2)}$$

$$\frac{n^2}{(n + \alpha)(n^2 - \beta^2)}$$

$$\frac{n^2}{(n + \alpha)(n^2 + \beta^2)}$$

$$\frac{n^2}{(n^3 + \alpha^3)}$$

APPENDIX A. TABLE OF PARTIAL FRACTIONS

A.1. $r=2$.

$$\checkmark \quad \frac{1}{n(n+\alpha)} = \frac{1}{\alpha} \left[\frac{1}{n} - \frac{1}{(n+\alpha)} \right]$$

$$\checkmark \quad \frac{1}{(n+\alpha)(n+\beta)} = \frac{1}{(\beta-\alpha)} \left[\frac{1}{(n+\alpha)} - \frac{1}{(n+\beta)} \right]$$

$$\checkmark \quad \frac{n}{(n+\alpha)(n+\beta)} = \frac{1}{(\alpha-\beta)} \left[\frac{\alpha}{(n+\alpha)} - \frac{\beta}{(n+\beta)} \right]$$

$$\checkmark \quad \frac{1}{(n^2-\alpha^2)} = \frac{1}{2\alpha} \left[\frac{1}{n-\alpha} - \frac{1}{n+\alpha} \right]$$

$$\checkmark \quad \frac{n}{(n^2-\alpha^2)} = \frac{1}{2} \left[\frac{1}{n-\alpha} + \frac{1}{n+\alpha} \right]$$

$$\checkmark \quad \frac{1}{(n^2+\alpha^2)} = \frac{1}{2i\alpha} \left[\frac{1}{n-i\alpha} - \frac{1}{n+i\alpha} \right]$$

$$\checkmark \quad \frac{n}{(n^2+\alpha^2)} = \frac{1}{2} \left[\frac{1}{n-i\alpha} + \frac{1}{n+i\alpha} \right]$$

A.2. $r=3$.

$$\checkmark \quad \frac{1}{n(n+\alpha)(n+\beta)} = \frac{1}{\alpha\beta(\alpha-\beta)} \left[\frac{(\alpha-\beta)}{n} + \frac{\beta}{(n+\alpha)} - \frac{\alpha}{(n+\beta)} \right]$$

$$\checkmark \quad \frac{1}{n^2(n+\alpha)} = \frac{1}{\alpha^2} \left[\frac{\alpha}{n^2} - \frac{1}{n} + \frac{1}{(n+\alpha)} \right]$$

$$\checkmark \quad \frac{1}{n(n+\alpha)^2} = \frac{1}{\alpha^2} \left[\frac{1}{n} - \frac{1}{(n+\alpha)} - \frac{\alpha}{(n+\alpha)^2} \right]$$

$$\checkmark \quad \frac{1}{n(n^2-\alpha^2)} = \frac{-1}{2\alpha^2} \left[\frac{2}{n} - \frac{1}{(n-\alpha)} - \frac{1}{(n+\alpha)} \right]$$

$$\checkmark \quad \frac{1}{n(n^2+\alpha^2)} = \frac{1}{2\alpha^2} \left[\frac{2}{n} - \frac{1}{(n-i\alpha)} - \frac{1}{(n+i\alpha)} \right]$$

$$\checkmark \quad \frac{n^k}{(n+\alpha)(n+\beta)(n+\gamma)} = \frac{1}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} \left[\begin{array}{l} + \frac{(-\alpha)^k(\beta-\gamma)}{(n+\alpha)} \\ - \frac{(-\beta)^k(\alpha-\gamma)}{(n+\beta)} \\ + \frac{(-\gamma)^k(\alpha-\beta)}{(n+\gamma)} \end{array} \right]$$

where: $0 \leq k \leq 2$

$$\checkmark \quad \frac{1}{(n+\alpha)(n+\beta)^2} = \frac{1}{(\alpha-\beta)^2} \left[\frac{1}{(n+\alpha)} - \frac{1}{(n+\beta)} + \frac{(\alpha-\beta)}{(n+\beta)^2} \right]$$

$$\checkmark \quad \frac{n}{(n+\alpha)(n+\beta)^2} = \frac{-1}{(\alpha-\beta)^2} \left[\frac{\alpha}{(n+\alpha)} - \frac{\alpha}{(n+\beta)} + \frac{\beta(\alpha-\beta)}{(n+\beta)^2} \right]$$

$$???\quad \frac{n^2}{(n+\alpha)(n+\beta)^2} = \frac{1}{(\alpha-\beta)^2} \left[\frac{\alpha^2}{(n+\alpha)} - \frac{\beta(2\alpha-\beta)}{(n+\beta)} + \frac{\beta^2(\alpha-\beta)}{(n+\beta)^2} \right]$$

$$\begin{aligned}
??? \frac{1}{(\beta^2 - \alpha^2)} ??? & \frac{1}{(n + \alpha)(n^2 - \beta^2)} = \frac{1}{2\beta(\alpha^2 - \beta^2)} \left[\frac{(\alpha + \beta)}{(n + \beta)} - \frac{(\alpha - \beta)}{(n - \beta)} - \frac{2\beta}{(n + \alpha)} \right] \\
??? \frac{1}{(\beta^2 - \alpha^2)} ??? & \frac{n}{(n + \alpha)(n^2 - \beta^2)} = \frac{-1}{2(\alpha^2 - \beta^2)} \left[\frac{(\alpha + \beta)}{(n + \beta)} + \frac{(\alpha - \beta)}{(n - \beta)} - \frac{2\alpha}{(n + \alpha)} \right] \\
??? \frac{1}{(\beta^2 - \alpha^2)} ??? & \frac{n^2}{(n + \alpha)(n^2 - \beta^2)} = \frac{1}{2(\alpha^2 - \beta^2)} \left[\frac{(\alpha + \beta)\beta}{(n + \beta)} - \frac{(\alpha - \beta)\beta}{(n - \beta)} - \frac{2\alpha^2}{(n + \alpha)} \right] \\
??? & \frac{1}{(n + \alpha)(n^2 + \beta^2)} = \frac{1}{2i\beta(\alpha^2 + \beta^2)} \left[\frac{(\alpha + i\beta)}{(n + \beta)} - \frac{(\alpha - i\beta)}{(n - \beta)} - \frac{2i\beta}{(n + \alpha)} \right] \\
??? & \frac{n}{(n + \alpha)(n^2 + \beta^2)} = \frac{-1}{2(\alpha^2 + \beta^2)} \left[\frac{(\alpha + i\beta)}{(n + \beta)} + \frac{(\alpha - i\beta)}{(n - \beta)} - \frac{2\alpha}{(n + \alpha)} \right] \\
??? & \frac{n^2}{(n + \alpha)(n^2 + \beta^2)} = \frac{1}{2i(\alpha^2 + \beta^2)} \left[\frac{(\alpha + i\beta)\beta}{(n + \beta)} - \frac{(\alpha - i\beta)\beta}{(n - \beta)} + \frac{2i\alpha^2}{(n + \alpha)} \right]
\end{aligned}$$

REFERENCES

- [1] Eldon R. Hansen, "A Table of Series and Products"
Prentice Hall (May 1975), ISBN-10: 0138819386, ISBN-13: 978-0138819385
- [2] This Paper. [math.NT]

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