GENERALIZATION OF HILBERT'S INEQUALITY

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Abstract. One the inequalities named Hilbert's inequality is,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{n+m} \leq \pi\left(\sum_{n=1}^{\infty}\|a_n\|^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}\|b_m\|^2\right)^{\frac{1}{2}} \tag{1}$$

This paper extends this Hilbert's inequality to the following more general inequalities:

$$\sum_{n\in S_1}\sum_{m\in S_2}\frac{a_n b_m}{r_1 n + r_2 m} \leq \pi P\left[\sum_{n\in S_1}\|a_n\|^2\right]^{\frac{1}{2}}\left[\sum_{m\in S_2}\|b_m\|^2\right]^{\frac{1}{2}} \tag{2a}$$

(2b)

$$\sum_{n\in S_1}\sum_{m\in S_2}\frac{a_n b_m}{(r_1 n + r_2 m)^2} \leq \frac{\pi^2 P^2}{3}\left[\sum_{n\in S_1}\|a_n\|^2\right]^{\frac{1}{2}}\left[\sum_{m\in S_2}\|b_m\|^2\right]^{\frac{1}{2}}$$

where:

- $r_1, r_2 \in \mathbb{Q}$ (posibly the same rational number),
- $r_1 n + r_2 m \neq 0$,
- P is the common period; the smallest, positive, rational number such that $r_1 P \in \mathbb{Z}$ and $r_2 P \in \mathbb{Z}$, and
- S_1 and S_1 are two finite or countably infinite sets of integers (possibly the same set)

The classic Hilbert inequality, (1), is a the special case of the generalized inequality, (2a), where $r_1 = r_2 = P = 1$.

The proofs generalizing the Hilbert inquality of (1) are simple extensions works of Jameson[1], Davenport, and Montgomery[2] who proved Hilbert's inequality using the Toeplitz method.

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1. Proof of First Extended Inequality

$$\sum_{n\in S_1}\sum_{m\in S_2}\frac{a_n b_m}{r_1 n + r_2 m} \leq \pi P\left[\sum_{n\in S_1}\|a_n\|^2\right]^{\frac{1}{2}}\left[\sum_{m\in S_2}\|b_m\|^2\right]^{\frac{1}{2}}$$

where:

- $r_1, r_2 \in \mathbb{Q}$ (posibly the same rational number),
- $r_1 n + r_2 m \neq 0$,
- P is the common period; the smallest, positive, rational number such that $r_1 P \in \mathbb{Z}$ and $r_2 P \in \mathbb{Z}$, and
- S_1 and S_1 are two finite or countably infinite sets of integers (possibly the same set)

Proof. There is no loss of generality in assuming the sequences a_n and b_m of equation (2a) are square-summable If one or both are not square-summable, then equation (2a) becomes:

$$\sum_{n\in S_1}\sum_{m\in S_2}\frac{a_n b_m}{r_1 n + r_2 m} \leq \infty$$

which is trivial true.

It is possible to construct three functions, $f_1(t)$, $A(t)$ and $B(t)$ which are defined as:

$$A(t) = \sum_{n\in S_1} a_n e^{2\pi i r_1 n t} \tag{3a}$$

$$B(t) = \sum_{m\in S_2} b_m e^{2\pi i r_2 m t} \tag{3b}$$

$$f_1(t) = (2\pi i)\left(t - \frac{P}{2}\right) \tag{3c}$$

The first half of the proof is to prove:

$$\sum_{n\in S_1}\sum_{m\in S_2}\frac{a_n b_m}{r_1 n + r_2 m} = \frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \tag{4}$$

where:

- $r_1, r_2 \in \mathbb{Q}$ (possibly the same rational number),
- $r_1 n + r_2 m \neq 0$,
- P is the smallest, positive, rational number such that $r_1 P \in \mathbb{Z}$ and $r_2 P \in \mathbb{Z}$, and
- S_1 and S_1 are two finite or countably infinite sets of integers (possibly the same set)

(5)
$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = \frac{1}{P}\int_{t=0}^{P}(2\pi i)\left(t-\frac{P}{2}\right)\sum_{n\in S_1} a_n e^{2\pi i r_1 n t}\sum_{m\in S_2} b_m e^{2\pi i r_2 m t}dt$$

Because the Fourier constants a_n and b_m are square-summable the convergence of the sums to $A(t)$ and $B(t)$, permit interchanging the summation and integration. Equation (5) then becomes:

(6)
$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\frac{2\pi i}{P}\int_{t=0}^{P}\left(t-\frac{P}{2}\right)e^{2\pi i r_1 n t}e^{2\pi i r_2 m t}dt\right]$$

(7)
$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\frac{2\pi i}{P}\int_{t=0}^{P}\left(t-\frac{P}{2}\right)e^{2\pi i t(r_1 n + r_2 m)}dt\right]$$

Two cases arise:

- $r_1 n + r_2 m = 0$ and
- $r_1 n + r_2 m \neq 0$.

For case 1, $r_1 n + r_2 m = 0$, equation (7) becomes:

(8) $$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\frac{2\pi i}{P}\int_{t=0}^{P}\left(t-\frac{P}{2}\right)dt\right]$$

(9) $$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m [0]$$

(10) $$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = 0$$

For case 2, $r_1 n + r_2 m \neq 0$, equation (7) becomes:

(11)
$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\frac{2\pi i}{P}\int_{t=0}^{P}\left(t-\frac{P}{2}\right)e^{2\pi i t(r_1 n + r_2 m)}dt\right]$$

(12)
$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\left(\frac{2\pi i}{P}\right)\left(\frac{P e^{2\pi i P(r_1 n + r_2 m)}}{2\pi i(r_1 n + r_2 m)}\right)\right]$$

Since, $r_2 P \in \mathbb{Z}$ and $r_2 P \in \mathbb{Z}$, it follows that for all n and m $(r_1 n + r_2 m)P \in \mathbb{Z}$ and $e^{2\pi i(r_1 n + r_2 m)P} = 1$. Using this, equation (12) can be re-written as:

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt = \sum_{n\in S_1}\sum_{m\in S_2}\frac{a_n b_m}{(r_1 n + r_2 m)} \tag{13}$$

The second half of the proof is to prove:

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \leq \pi P\left[\sum_{n\in S_1}\|a_n\|^2\right]^{\frac{1}{2}}\left[\sum_{m\in S_2}\|b_m\|^2\right]^{\frac{1}{2}} \tag{14}$$

where:

- $r_1, r_2 \in \mathbb{Q}$ (possibly the same rational number),
- $r_1 n + r_2 m \neq 0$,
- P is the smallest, positive, rational number such that $r_1 P \in \mathbb{Z}$ and $r_2 P \in \mathbb{Z}$, and
- S_1 and S_1 are two finite or countably infinite sets of integers (possibly the same set)

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \leq \int_{t=0}^{P}\left|\frac{1}{P}f_1(t)\right||A(t)B(t)|\,dt \tag{15}$$

$$\left|\frac{1}{P}f_1(t)\right| = \left|\left(\frac{1}{P}\right)(2\pi i)\left(t - \frac{P}{2}\right)\right| \leq \left(\frac{1}{P}\right)(2\pi)\left(\frac{P}{2}\right) = \pi$$

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \leq \pi\int_{t=0}^{P}|A(t)B(t)|\,dt \tag{16}$$

By the Cauchy-Schwartz inequality:

$$\int_{t=0}^{P}|A(t)B(t)|\,dt \leq \left[\int_{t=0}^{P}\|A(t)\|^2 dt\right]^{\frac{1}{2}}\left[\int_{t=0}^{P}\|B(t)\|^2 dt\right]^{\frac{1}{2}} \tag{17}$$

And by Parseval's Theorem

$$\int_{t=0}^{P}\|A(t)\|^2 dt = P\sum_{n\in S_1}\|a_n\|^2 \tag{18a}$$

$$\int_{t=0}^{P}\|B(t)\|^2 dt = P\sum_{m\in S_2}\|b_m\|^2 \tag{18b}$$

Combining (17) and (18) with (16) yields:

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \le \pi \left[P\sum_{n\in S_1}\|a_n\|^2\right]^{\frac{1}{2}} \left[P\sum_{m\in S_2}\|b_m\|^2\right]^{\frac{1}{2}}$$

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \le \pi P \left[\sum_{n\in S_1}\|a_n\|^2\right]^{\frac{1}{2}} \left[\sum_{m\in S_2}\|b_m\|^2\right]^{\frac{1}{2}} \tag{19}$$

Combining the two halves of the proof, (13) and (19), yields:

$$\sum_{n\in S_1}\sum_{m\in S_2}\frac{a_n b_m}{r_1 n + r_2 m} \le \pi P \left[\sum_{n\in S_1}\|a_n\|^2\right]^{\frac{1}{2}} \left[\sum_{m\in S_2}\|b_m\|^2\right]^{\frac{1}{2}} \tag{20}$$

□

2. Proof of Second Extended Inequality

$$\sum_{n\in S_1}\sum_{m\in S_2}\frac{a_n b_m}{(r_1 n + r_2 m)^2} \leq \left(\frac{\pi^2 P^2}{3}\right)\left[\sum_{n\in S_1}\|a_n\|^2\right]^{\frac{1}{2}}\left[\sum_{m\in S_2}\|b_m\|^2\right]^{\frac{1}{2}}$$

where:

- $r_1, r_2 \in \mathbb{Q}$ (posibly the same rational number),
- $r_1 n + r_2 m \neq 0$,
- P is the common period; the smallest, positive, rational number such that $r_1 P \in \mathbb{Z}$ and $r_2 P \in \mathbb{Z}$, and
- S_1 and S_1 are two finite or countably infinite sets of integers (possibly the same set)

Proof. The proof of the second inequality follows the form of the proof above:

(1) Select appropriate definitions of $f(t)$, $A(t)$, and $B(t)$.
(2) Show that the summation to the left of the inequality is equal to the integral: $\int f(t)A(t)B(t)dt$.
(3) Show the integral is less than or equal to the sum of Fourier constants on the right of the inequality.

It is possible to construct three functions, $f_2(t)$, $A(t)$ and $B(t)$ which are defined as:

$$A(t) = \sum_{n\in S_1} a_n e^{2\pi i r_1 n t} \tag{21a}$$

$$B(t) = \sum_{m\in S_2} b_m e^{2\pi i r_2 m t} \tag{21b}$$

$$f_2(t) = 2\pi^2\left(t^2 - Pt + \frac{P^2}{6}\right) \tag{21c}$$

(22)
$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = \frac{1}{P}\int_{t=0}^{P} 2\pi^2\left(t^2 - Pt + \frac{P^2}{6}\right)\sum_{n\in S_1} a_n e^{2\pi i r_1 n t}\sum_{m\in S_2} b_m e^{2\pi i r_2 m t}dt$$

(23)
$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\frac{2\pi^2}{P}\int_{t=0}^{P}\left(t^2 - Pt + \frac{P^2}{6}\right)e^{2\pi i r_1 n t}e^{2\pi i r_2 m t}dt\right]$$

(24)
$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\frac{2\pi^2}{P}\int_{t=0}^{P}\left(t^2 - Pt + \frac{P^2}{6}\right)e^{2\pi i t(r_1 n + r_2 m)}dt\right]$$

In proving (24), two case again arise:

- $r_1 n + r_2 m = 0$ and
- $r_1 n + r_2 m \neq 0$.

For the case where $r_1 n + r_2 m = 0$, equation (24) becomes:
(25)
$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\frac{2\pi^2}{P}\int_{t=0}^{P}\left(t^2 - Pt + \frac{P^2}{6}\right)dt\right]$$

$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \,[0] \tag{26}$$

$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = 0 \tag{27}$$

For case 2, $r_1 n + r_2 m \neq 0$, equation (24) becomes:
(28)
$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\frac{2\pi^2}{P}\int_{t=0}^{P}\left(t^2 - Pt + \frac{P^2}{6}\right)e^{2\pi i t(r_1 n + r_2 m)}dt\right]$$

(29)
$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = \sum_{n\in S_1} a_n \sum_{m\in S_2} b_m \left[\left(\frac{2\pi^2}{P}\right)\left(\frac{Pe^{2\pi i P(r_1 n + r_2 m)}}{2\pi^2 (r_1 n + r_2 m)^2}\right)\right]$$

Since, $r_2 P \in \mathbb{Z}$ and $r_2 P \in \mathbb{Z}$, it follows that for all n and m $(r_1 n + r_2 m)P \in \mathbb{Z}$ and $e^{2\pi i(r_1 n + r_2 m)P} = 1$. Using this, equation (29) can be re-written as:

$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt = \sum_{n\in S_1}\sum_{m\in S_2} \frac{a_n b_m}{(r_1 n + r_2 m)^2} \tag{30}$$

Again, the second half is to prove the integral is less than the sum of the Fourier constants.

$$\frac{1}{P}\int_{t=0}^{P} f_2(t)A(t)B(t)dt \leq \int_{t=0}^{P}\left|\frac{1}{P}f_2(t)\right| |A(t)B(t)|\, dt \tag{31}$$

$$\left|\frac{1}{P}f_2(t)\right| = \left|\left(\frac{1}{P}\right)(2\pi^2)\left(t^2 - Pt + \frac{P^2}{6}\right)\right| \le \left|\left(\frac{2\pi^2}{P}\right)\left(\frac{P^2}{6}\right)\right| = \frac{\pi^2 P}{3}$$

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \le \left(\frac{\pi^2 P}{3}\right)\int_{t=0}^{P} |A(t)B(t)|\, dt \tag{32}$$

Again, by the Cauchy-Schwartz inequality:

$$\int_{t=0}^{P} |A(t)B(t)|\, dt \le \left[\int_{t=0}^{P} \|A(t)\|^2 dt\right]^{\frac{1}{2}} \left[\int_{t=0}^{P} \|B(t)\|^2 dt\right]^{\frac{1}{2}} \tag{33}$$

And again by Parseval's Theorem

$$\int_{t=0}^{P} \|A(t)\|^2 dt = P \sum_{n\in S_1} \|a_n\|^2 \tag{34a}$$

$$\int_{t=0}^{P} \|B(t)\|^2 dt = P \sum_{m\in S_2} \|b_m\|^2 \tag{34b}$$

Combining (33) and (34) with (32) yields:

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \le \left(\frac{\pi^2 P}{3}\right)\left[P\sum_{n\in S_1} \|a_n\|^2\right]^{\frac{1}{2}} \left[P\sum_{m\in S_2} \|b_m\|^2\right]^{\frac{1}{2}}$$

$$\frac{1}{P}\int_{t=0}^{P} f_1(t)A(t)B(t)dt \le \left(\frac{\pi^2 P^2}{3}\right)\left[\sum_{n\in S_1} \|a_n\|^2\right]^{\frac{1}{2}} \left[\sum_{m\in S_2} \|b_m\|^2\right]^{\frac{1}{2}} \tag{35}$$

Combining the two halves of the proof, (30) and (35), yields:

$$\sum_{n\in S_1}\sum_{m\in S_2} \frac{a_n b_m}{r_1 n + r_2 m} \le \left(\frac{\pi^2 P^2}{3}\right)\left[\sum_{n\in S_1} \|a_n\|^2\right]^{\frac{1}{2}} \left[\sum_{m\in S_2} \|b_m\|^2\right]^{\frac{1}{2}} \tag{36}$$

□

References

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